3

Grade-Level Considerations

Implementation of the standards will be challenging, especially during the early phases, when many students will not have the necessary foundational skills to master all of the expected grade-level mathematics content. This chapter provides a discussion of the mathematical considerations that went into the selection of the individual standards and describes the major roles some of them play in a standards-based curriculum. It also indicates areas where students may have difficulties, and, when possible, it provides techniques for easing them. Finally, it points out subtleties to which particular attention must be paid.

The chapter includes the following categories for each of the earlier grades:

- Areas of emphasis—Targets key areas of learning (These are taken directly from the *Mathematics Content Standards.*)
- Key standards—Identifies (⬬) some of the most important standards and tries to place them into context
- Elaboration—Provides added detail on these standards and on a number of related ones
- Grade-level accomplishments—Identifies areas of mathematics readiness and learning that are likely to present particular difficulties and concerns

The five strands in the *Mathematics Content Standards* (Number Sense; Algebra and Functions; Measurement and Geometry; Statistics, Data Analysis, and Probability; and Mathematical Reasoning) organize information about the key standards for kindergarten through grade seven. **It should be noted that the strand of mathematical reasoning is different from the other four strands.** This strand, which is inherently embedded in each of the other strands, is fundamental in developing the basic skills and conceptual understanding for a solid mathematical foundation. **It is important when looking at the standards to see the reasoning in all of them.** Since this is the case, this chapter does not highlight key topics in the Mathematical Reasoning strand.

Mathematical reasoning is inherently embedded in each of the other strands.

The section for grades eight through twelve in this chapter is organized by discipline, and only the basic ones—Algebra I; geometry; Algebra II; trigonometry; the precalculus course, mathematical analysis; and probability and statistics—are discussed in detail. The remaining courses are guided by other considerations, such as the *Advanced Placement (AP)* tests, and are outside the scope of this document.

The grade-level readiness information, which relates to difficult content areas in mathematics, is relevant to all teachers, students, and classrooms. This information will be particularly helpful in determining whether students need to be provided with specific intervention materials and additional instruction to learn the grade-level mathematics.

The Strands

The content of the mathematics curriculum has frequently been divided into categories called strands. Like most systems of categories, the strands in mathematics were developed to break the content into a small set of manageable and understandable categories. Since there is no universal agreement on the selection of the parts, the use of strands is somewhat artificial; and many different systems have been suggested. In addition, it is often difficult to restrict a particular mathematical concept or skill to a single strand. Nonetheless, this framework continues the practice of presenting the content of mathematics in five strands for kindergarten through grade seven.

Because the content of mathematics builds and changes from grade to grade, the content in any one strand changes considerably over the course of mathematics programs for kindergarten through grade seven. Thus the strands serve only as an aid to organizing and thinking about the curriculum but no more than that. They describe the curriculum rather than define it. For the same reason the identification of strands does not mean that each is to be given equal weight in each year of mathematics education.

The general nature of each strand is described in the sections that follow.

Number Sense

Much of school mathematics depends on numbers, which are used to count, compute, measure, and estimate. The mathematics for this standard centers primarily on the development of number concepts; on computation with numbers (addition, subtraction, multiplication, division, finding powers and roots, and so forth); on numeration (systems for writing numbers, including base ten, fractions, negative numbers, rational numbers, percents, scientific notation, and so forth); and on estimation. At higher levels this strand includes the study of prime and composite numbers, of irrational numbers and their approximation by rationals, of real numbers, and of complex numbers.

Algebra and Functions

This strand involves two closely related subjects. Functions are rules that assign to each element in an initial set an element in a second set. For example, as early as kindergarten, children take collections of colored balls and sort them according to color, thereby assigning to each ball its color in the process. Later, students work with simple numeric functions, such as unit conversions that assign quantities of measurement; for example, 12 inches to each foot.

Functions are, therefore, one of the key areas of mathematical study. As indicated, they are encountered informally in the elementary grades and grow in prominence and importance with the student's increasing grasp of algebra in the higher grades. Beginning with the first year of algebra, functions are encountered at every turn.

Functions are one of the key areas of mathematical study.

Algebra proper again starts informally. It appears initially in its proper form in the third grade as "generalized arithmetic." In later grades algebra is the vital tool needed for solving equations and inequalities and using them as mathematical models of real situations. Students solve the problems that arise by translating from natural language—by which they communicate daily—to the abstract language of algebra and, conversely, from the formal language of algebra to natural language to demonstrate clear understanding of the concepts involved.

Measurement and Geometry

Geometry is the study of space and figures in space. In school any study of space, whether practical or theoretical, is put into the geometry strand. In the early grades this strand includes the use of measuring tools, such as rulers, and recognition of basic shapes, such as triangles, circles, squares, spheres, and cubes. In the later grades the content extends to the study of area and volume and the measurement of angles. In high school, plane geometry is studied both as an introduction to the concept of mathematical proof and as a fascinating structure that has profoundly influenced civilization for more than 2,000 years.

Statistics, Data Analysis, and Probability

This strand includes the definitions and calculations of various averages and the analysis of data by classification and by graphical displays, taking into account randomness and bias in sampling. This strand has important connections with Strand 2, Algebra and Functions, and Strand 1, Number Sense, in the study of permutations and combinations and of Pascal's triangle. In the elementary grades effort is largely limited to collecting data and displaying it in graphs, in addition to calculating simple averages and performing probability experiments. This strand becomes more important in grade seven and above, when the students have gained the necessary skill with fractions and algebraic concepts in general so that statistics and their impact on daily life can be discussed with more sophistication than would have been possible earlier.

Mathematical Reasoning

Whenever a mathematical statement is justified, mathematical reasoning is involved. Mathematical reasoning in an inductive form appears in the early grades and is soon joined by deductive reasoning. Mathematical reasoning is involved in explaining arithmetic facts, in solving problems and puzzles at all levels, in understanding algorithms and formulas, and in justifying basic results in all areas of mathematics.

Mathematical reasoning, requiring careful, concise, and comprehensible proofs, is at the heart of mathematics and, indeed, is the essence of the discipline, differentiating it from others. Students must realize that assumptions are always involved in reaching conclusions, and they must recognize when assumptions are being made. Students must develop the habits of logical thinking and of recognizing and critically questioning all assumptions. In later life such reasoning skills will provide students with a foundation for making sound decisions and give them an invaluable defense against misleading claims.

Mathematical reasoning, requiring careful, concise, and comprehensible proofs, is at the heart of mathematics.

Key Standards

Grade	Number Sense	Algebra and Functions	Measurement and Geometry	Statistics, Data Analysis, and Probability	Mathematical Reasoning*
Kindergarten	**1.0** 1.1 1.2 1.3 2.0 **2.1** 3.0 3.1	1.0 **1.1**	**1.0** 1.1 1.2 1.3 1.4 2.0 2.1 2.2	1.0 1.1 **1.2**	1.0 1.1 1.2 2.0 2.1 2.2
Grade One	1.0 **1.1** **1.2** 1.3 1.4 1.5 2.0 **2.1** **2.2** **2.3** **2.4** **2.5** 2.6 2.7 3.0 3.1	1.0 1.1 1.2 1.3	1.0 1.1 1.2 2.0 2.1 2.2 2.3 2.4	1.0 1.1 1.2 2.0 **2.1**	1.0 1.1 1.2 2.0 2.1 2.2 3.0
Grade Two	1.0 **1.1** 1.2 **1.3** 2.0 **2.1** **2.2** 2.3 **3.0** **3.1** **3.2** **3.3** 4.0 **4.1** **4.2** **4.3** 5.0 **5.1** **5.2** 6.0 6.1	1.0 **1.1** 1.2 1.3	1.0 1.1 1.2 **1.3** 1.4 1.5 **2.0** **2.1** **2.2**	**1.0** 1.1 1.2 1.3 1.4 **2.0** 2.1 2.2	1.0 1.1 1.2 2.0 2.1 2.2 3.0
Grade Three	1.0 1.1 1.2 **1.3** 1.4 **1.5** 2.0 **2.1** **2.2** **2.3** **2.4** 2.5 2.6 2.7 2.8 3.0 3.1 **3.2** **3.3** 3.4	1.0 **1.1** 1.2 1.3 1.4 1.5 2.0 **2.1** 2.2	1.0 1.1 **1.2** **1.3** 1.4 2.0 **2.1** **2.2** **2.3** 2.4 2.5 2.6	1.0 1.1 **1.2** **1.3** 1.4	1.0 1.1 1.2 2.0 2.1 2.2 2.3 2.4 2.5 2.6 3.0 3.1 3.2 3.3

***It should be noted that the strand of mathematical reasoning is different from the other four strands.** This strand, which is inherently embedded in each of the other strands, is fundamental in developing the basic skills and conceptual understanding for a solid mathematical foundation. It is important when looking at the standards to see the reasoning in all of them. Since this is the case, the key topics in the mathematical reasoning strand are not highlighted. Standards with the ⬬ symbol are the most important ones to be covered within a grade level.

Key Standards

	Number Sense	Algebra and Functions	Measurement and Geometry	Statistics, Data Analysis, and Probability	Mathematical Reasoning
Grade Four	1.0 **1.1** **1.2** **1.3** **1.4** 1.5 1.6 1.7 **1.8** **1.9** 2.0 2.1 2.2 **3.0** **3.1** **3.2** **3.3** **3.4** 4.0 4.1 **4.2**	1.0 1.1 **1.2** **1.3** 1.4 **1.5** **2.0** **2.1** **2.2**	1.0 1.1 1.2 1.3 1.4 **2.0** **2.1** **2.2** **2.3** 3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8	1.0 1.1 1.2 1.3 2.0 2.1 2.2	1.0 1.1 1.2 2.0 2.1 2.2 2.3 2.4 2.5 2.6 3.0 3.1 3.2 3.3
Grade Five	1.0 1.1 **1.2** 1.3 **1.4** **1.5** 2.0 **2.1** **2.2** **2.3** 2.4 2.5	1.0 1.1 **1.2** 1.3 **1.4** **1.5**	1.0 **1.1** **1.2** **1.3** 1.4 2.0 **2.1** **2.2** 2.3	1.0 1.1 1.2 1.3 **1.4** **1.5**	1.0 1.1 1.2 2.0 2.1 2.2 2.3 2.4 2.5 2.6 3.0 3.1 3.2 3.3
Grade Six	**1.0** **1.1** **1.2** **1.3** **1.4** **2.0** 2.1 2.2 **2.3** **2.4**	1.0 **1.1** 1.2 1.3 1.4 2.0 2.1 **2.2** 2.3 3.0 3.1 3.2	1.0 **1.1** 1.2 1.3 2.0 2.1 **2.2** 2.3	1.0 1.1 1.2 1.3 1.4 2.0 2.1 **2.2** **2.3** **2.4** **2.5** 3.0 **3.1** 3.2 **3.3** 3.4 **3.5**	1.0 1.1 1.2 1.3 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 3.0 3.1 3.2 3.3
Grade Seven	1.0 1.1 **1.2** 1.3 **1.4** **1.5** 1.6 **1.7** 2.0 2.1 **2.2** **2.3** 2.4 **2.5**	1.0 1.1 1.2 **1.3** 1.4 1.5 2.0 2.1 2.2 3.0 3.1 3.2 **3.3** **3.4** **4.0** **4.1** **4.2**	1.0 1.1 1.2 **1.3** 2.0 2.1 2.2 2.3 2.4 3.0 3.1 3.2 **3.3** **3.4** 3.5 **3.6**	1.0 1.1 1.2 **1.3**	1.0 1.1 1.2 1.3 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 3.0 3.1 3.2 3.3

Preface to Kindergarten Through Grade Seven

Mathematics, in the kindergarten through grade seven curriculum, starts with basic material and increases in scope and content as the years progress. It is like an inverted pyramid, with the entire weight of the developing subject resting on the core provided in kindergarten through grade two, when numbers, sets, and functions are introduced. If the introduction of the subject in the early grades is flawed, then later on, students can have extreme difficulty progressing; and their mathematical development can stop prematurely, leaving them, in one way or another, unable to fully realize their potential.

Because the teaching of mathematics in the early grades is largely synonymous with the problems given to the students, it is essential that students be presented with carefully constructed and mathematically accurate problems throughout their school careers. Problems which appear correct can actually be wrong, leading to serious misunderstandings on the part of the students. For example, the teacher might present the kindergarten standard for Algebra and Functions 1.1: "Identify, sort, and classify objects by attribute and identify objects that do not belong to a particular group." At first glance, the following exercise might seem appropriate for this standard:

It is essential that students be presented with carefully constructed and mathematically accurate problems throughout their school careers.

> *A picture of three objects, a basketball, a bus, and a tennis ball, is shown to the students, and they are asked to tell which one does not belong.*

This statement appears to present a perfectly reasonable problem. The difficulty is that, as stated, the question is not a problem in mathematics. From a mathematical point of view, the question is to determine which of these objects belongs to one set while the third belongs to a different one. It must be clear that unless the sets are specified in some way, the question cannot have a reasonable answer. In this case, the student must *guess* that the teacher is asking the student to sort objects by shape. The following might be asked instead: *We want to collect balls. Which of these objects should we select?* Or perhaps the contrapositive*, Which of these objects should not be included?* Another approach is to add colors; for example, coloring the bus and tennis ball blue and the basketball brown. Then a different question might be asked*: We want blue things. Which of these objects do we want?* or *We want round, blue objects. Which of these do we want?* But a question in the mathematics part of the curriculum should not be asked when the assumptions underlying what is wanted are not clearly stated.

In another example, the standard for Statistics, Data Analysis, and Probability 1.2 asks students to identify, describe, and extend simple patterns involving shape, size, or color, such as a circle or triangle or red or blue. A possible problem illustrating the standard follows:

> *The students are given a picture that shows in succession a rectangle, triangle, square, rectangle, triangle, square, blank, triangle, square. The students are asked to fill in the blank.*

While this problem may seem to be a reasonable one (and an example of problems that all too commonly appear in the mathematics curricula of the lower grades), it cannot be considered a problem in mathematics. From a mathematical point of view, there is no correct answer to this problem unless more data are supplied to the students. Mathematics is about drawing logical conclusions from explicitly stated hypotheses. *Because there is no statement about the nature of the pattern in this case (e.g., does the pattern repeat itself every three terms? every seven terms? every nine terms?), students can only guess at what should be in the blank spot.*

The intent of the problem was probably to ask students to infer from the given data that the pattern, in all likelihood, repeats itself every three terms, leaving students to assume that a rectangle belongs in the blank spot. But if students were to start thinking that every mathematical situation always contains a hidden agenda for them to guess correctly before they can proceed, then both the teaching and learning of mathematics would be tremendously compromised. Observations from some university-level mathematicians suggest that this outcome may have already occurred with some students. Students' reluctance to take mathematical statements at face value has become a major stumbling block.

If students were to start thinking that every mathematical situation always contains a hidden agenda for them to guess correctly before they can proceed, then both the teaching and learning of mathematics would be tremendously compromised.

In an attempt to make mathematics "more relevant," problems described as "real world" are often introduced. The following example of such a problem is similar to many fourth grade assessment problems: *The picture below shows a 5×5 section of an array of lockers with only the 3×3 center group numbered.*

	11	12	13	
	20	21	22	
	29	30	31	

Figure 1

Students are given the following assessment task: *Some of the numbers have fallen off the doors of some old lockers. Figure out the missing numbers and describe the number pattern.*

This problem does not make sense mathematically. The data given are insufficient to find a unique answer. In fact, the expected "solution," as shown in figure 2, makes use of the *hidden assumption* that the array was rectangular. However, the assumptions that are given do not indicate that this is the case, and it would be improper, mathematically, to also assume that the array is rectangular.

1	2	3	4	5
10	11	12	13	14
19	20	21	22	23
28	29	30	31	32
37	38	39	40	41

Figure 2

There are many other solutions without this assumption. For example, one is shown in figure 3.

One of the key points of mathematics is to promote critical thinking. Students have to learn to reason precisely with the data given so that if assumptions are hidden, they know they must seek them out and question them.

One of the key points of mathematics is to promote critical thinking.

These remarks are not meant to diminish the importance of learning the number system and basic arithmetic, both of which are crucial as well. Here, too, these topics present problems for the kindergarten through grade seven curriculum, but not to the same degree as in many of the other areas discussed previously.

The intent of the material that follows in this chapter is to try to place into correct perspective much of the material taught in these grades, to indicate where problems might be encountered with some of the most important of these topics, and to suggest some ways of resolving the difficulties. In addition, throughout this chapter some items are pointed out to show where careful development will help foster critical thinking, and suggestions are given for accomplishing this process.

1	2	3	4	5	6	7				8	9
10	11	12	13	14	15	16				17	18
19	20	21	22	23	24	25				26	27
28	29	30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49	50	51

Figure 3

Kindergarten Areas of Emphasis

By the end of kindergarten, students understand small numbers, quantities, and simple shapes in their everyday environment. They count, compare, describe, and sort objects and develop a sense of properties and patterns.

Number Sense

1.0 1.1 1.2 1.3

2.0 **2.1**

3.0 3.1

Algebra and Functions

1.0 **1.1**

Measurement and Geometry

1.0 1.1 1.2 1.3 1.4

2.0 2.1 2.2

Statistics, Data Analysis, and Probability

1.0 1.1 **1.2**

Mathematical Reasoning

1.0 1.1 1.2

2.0 2.1 2.2

Key Standards

Number Sense

The Number Sense standard that follows is basic in kindergarten:

1.0 Students understand the relationship between numbers and quantities (i.e., that a set of objects has the same number of objects in different situations regardless of its position or arrangement).

A key skill within this standard is to group and compare sets of concrete items and recognize whether there are more, fewer, or an equal number of items in different sets.

The following Number Sense standard is also important:

2.1 Use concrete objects to determine the answers to addition and subtraction problems (for two numbers that are each less than 10).

The object of these standards is to begin to develop a precise sense of what a number is. Although students at this stage are dealing mainly with small numbers, they also need experience with larger numbers. An activity to provide this experience is to have the teacher fill glass jars with tennis balls, ping-pong balls, or jelly beans and ask the students to guess how many of these items are in the glass jar. Activities such as this one help give students an understanding of magnitude of numbers and help them gain experience with estimation.

When presenting this activity, teachers need to be aware that students can get the misconception that large numbers are only approximate rather than corresponding to exact quantities. This is a serious problem that has the potential to cause real difficulty later.

One way of avoiding this difficulty is to have the students use manipulatives, such as blocks, to compare two (relatively) large numbers; for example, 14 and 15. The class can explore the fact that 14 breaks up into two equal groups of 7, while 15 cannot be broken into two equal groups. The students would begin to appreciate that although visually distinguishing 15 objects from 14 without careful counting is difficult, the two numbers, nonetheless, are quite different. This activity should help students develop an awareness that each whole number is unique and will help them meet Number Sense Standard 1.2, which requires them to count and represent objects up to 30.

Algebra and Functions

The role of the Algebra and Functions standard is also basic:

1.1 Identify, sort, and classify objects by attribute and identify objects that do not belong to a particular group (e.g., all these balls are green, those are red).

Although kindergarten teachers may not think of themselves as algebra teachers, they actually begin the process. They make students aware of the existence of patterns by giving them their first experience of finding them in data, by providing their initial exposure to functions, and by introducing them to abstraction. For example, students realize that a blue rectangular block and a blue ball, which obviously have different physical attributes, can nevertheless be sorted together because of their common color. This realization is the beginning of abstract reasoning, which is a higher-order thinking skill.

Although kindergarten teachers may not think of themselves as algebra teachers, they actually begin the process.

Statistics, Data Analysis, and Probability

This standard interacts with the following Statistics, Data Analysis, and Probability standard:

1.2 Identify, describe, and extend simple patterns (such as circles or triangles) by referring to their shapes, sizes, or colors.

Kindergarten

Elaboration

The kindergarten teacher is likely to find that many students can learn more material than is specified in the kindergarten standards. For example, the standard for committing addition and subtraction facts to memory appears in the first grade. Because committing facts to memory requires substantial amounts of practice over an extended period, memorizing addition and subtraction facts can begin in kindergarten with simple facts, such as +1s, +2s, –1s, or sums to 10. Any practice of addition and subtraction facts should be limited to these more simple problems. Likewise, students can be taught the meaning of the symbols +, –, and = in the context of addition or subtraction, but again the focus is on small numbers. In measurement, the months can be taught in kindergarten as students learn the days of the week.

Considerations for Grade-Level Accomplishments in Kindergarten

Kindergarten is a critical time for children who, when they enter school, are behind their peers in the acquisition of skills and concepts. Efficient teaching in kindergarten can help prepare these children to work at an equal level with their peers in the later grades.

Students who enter kindergarten without some background in academic language (the language of tests and texts) and an understanding of the concepts such language represents have a great disadvantage in learning mathematics. Critical for beginning mathematical development are attributes, such as color, shape, and size; abstract concepts, such as *some, all,* and *none;* and ordinal concepts, such as *before, after, yesterday,* and *tomorrow*. Teachers need clear directions on how to maximize progress in mathematics for students with limited understanding of language concepts or for students who know the concepts in their native language but do not yet know the English words for them. Kindergarten provides many opportunities for teachers to teach basic mathematics vocabulary and concepts during instructional time or playtime; for example, students learn to take turns during a game or line up for recess (first, second, third), count off in a line (one, two, three), or determine the number of children who can take six balls out for recess if each child gets a ball (matching sets).

Kindergarten provides many opportunities for teachers to teach basic mathematics vocabulary and concepts.

The most important mathematical skills and concepts for children in kindergarten to acquire are described as follows:

- **Counting.** Before beginning instruction in counting, teachers should determine the number to which the child can already count and whether the child understands what each number represents. The teacher models the next few numbers in the sequence (e.g., 5, 6, 7); provides practice for the children in saying the counting sequence through the new numbers (1, 2, 3, 4, 5, 6, 7); and matches each number to a corresponding set of objects. After a student has mastered the sequence including the new numbers, the teacher introduces several more numbers and follows the same procedure. Even though the standard requires a mastery of counting only to 30, daily practice in counting can be provided until students can count to 50 or 100 so that they may be better prepared for the challenges of the first grade.

- **Reading numerals.** The teacher should introduce numerals after the children can count to 10. Confusion between numeral names and the counting order can be *decreased* if the teacher does not introduce the numerals in order. For example, the teacher introduces the numeral 4 and then 7. For several days the teacher introduces a new numeral until the students can identify the numerals 1 through 10. The teacher should provide cumulative practice by having students review previously introduced numbers while he or she presents a new number.
- **Writing numerals.** The standards require that students know the names of the numerals from 1 to 9 and how to write them. Generally, writing numbers will require a good deal of practice; and at this age some children may have difficulty with coordination. First, students should copy a numeral many times. Then they should write it with some prompts (e.g., dots or arrows); and later they should write it from memory, with the teacher saying the number and the student writing the numeral. A multisensory approach is very important here. Teachers must encourage the students of this age not to be concerned about the quality of their handwriting as they write numerals. Young children do not yet have fully developed fine-motor skills. Many students become frustrated by the discrepancy between what they want to produce on paper and what they can actually produce.
- **Understanding place value—reading numbers in the teens.** To read and write numbers from 10 to 20, students will need to understand something about place value. The teacher can expect the numbers 11, 12, 13, and 15 to be more troublesome than 14, 16, 17, 18, and 19. The second group is regular in pronunciation (e.g., *four*teen, *six*teen), but the first group is irregular; twelve is not pronounced as "twoteen" but as "twelve."

  An important prerequisite for understanding place value is being able to answer fact questions verbally; for example, what is 10 + 6? When the students know the facts about numbers in the teens that are regular in pronunciation, the teacher can introduce one number with irregular pronunciation and mix it with the regular numbers in a verbal exercise. New irregular numbers can be introduced as students demonstrate knowledge of previously introduced facts about numbers in the teens. Reading and writing these numbers can be introduced when students are able to do the verbal exercises.

An important prerequisite for understanding place value is being able to answer fact questions verbally.

- **Learning the days of the week.** The days of the week can be taught in a manner similar to that for counting, in which the teacher models a part of the sequence of days (Monday, Tuesday, Wednesday); provides practice in saying the sequence; introduces a new part after several days (Thursday, Friday); provides practice with this part; and then repeats the sequence from the beginning. The months of the year can also be taught in kindergarten. Unless the students have a firm understanding of the sequence of days and months, they will have difficulty with items applying concepts of time, such as *before* and *after* as indicated in the second part of the following standard:

Measurement and Geometry

1.0 Students understand the concept of time and units to measure it; they understand that objects have properties, such as length, weight, and capacity, and that comparisons may be made by referring to those properties.

Grade One Areas of Emphasis

By the end of grade one, students understand and use the concept of ones and tens in the place value number system. Students add and subtract small numbers with ease. They measure with simple units and locate objects in space. They describe data and analyze and solve simple problems.

Number Sense

1.0 **1.1** **1.2** 1.3 1.4 1.5
2.0 **2.1** **2.2** **2.3** **2.4** **2.5** 2.6 2.7
3.0 3.1

Algebra and Functions

1.0 1.1 1.2 1.3

Measurement and Geometry

1.0 1.1 1.2
2.0 2.1 2.2 2.3 2.4

Statistics, Data Analysis, and Probability

1.0 1.1 1.2
2.0 **2.1**

Mathematical Reasoning

1.0 1.1 1.2
2.0 2.1 2.2
3.0

Key Standards

Number Sense

The following Number Sense standard is basic:

1.1 Count, read, and write whole numbers to 100.

It is important that students gain a conceptual understanding of numbers and counting, not simply learn to count to 100 by rote. They need to understand, for example, that counting can occur in any order and in any direction, not just in the standard left-to-right counting pattern, as long as each item is tagged once and only once. Students must understand that numbers represent sets of specific

Grade One

quantities of items. Of particular importance is learning and understanding the counting sequence for numbers in the teens and multiples of ten. It should be emphasized that numbers in the teens represent a ten value and a certain number of unit values—12 does not merely represent a set of 12 items; it also represents 1 ten and 2 ones. A related and equally important Number Sense standard is:

1.2 Compare and order whole numbers to 100 by using the symbols for less than, equal to, or greater than ($<$, $=$, $>$).

The continuing development of addition and subtraction skills as described in the following standards is basic:

2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.

2.5 Show the meaning of addition (putting together, increasing) and subtraction (taking away, comparing, finding the difference).

For example, students should understand that the equation $15 - 8 = 7$ is the same as $15 = 7 + 8$. Particular attention should be paid to the assessment of these competencies because students who fail to learn these topics will have serious difficulties in the later grades. The achievement of these standards will require that students be exposed to and asked to solve simple addition and subtraction problems throughout the school year.

Statistics, Data Analysis, and Probability

The following Statistics, Data Analysis, and Probability standard is also important, but it has to be handled carefully:

2.1 Describe, extend, and explain ways to get to a next element in simple repeating patterns (e.g., rhythmic, numeric, color, and shape).

Students should *never* get the idea that the next term *automatically* repeats (unless they are told explicitly that it does); however, it is legitimate to ask what is the *most likely* next term. In this way students begin to learn not only the usefulness of patterns in sorting and understanding data but also careful, precise patterns of thought. Examples are sequences of colors, such as red, blue, red, blue, . . . or numbers, 1, 2, 3, 1, 2, 3, 1, 2, 3, . . . But more complex series might also be used, such as 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, . . .

Elaboration

Teaching students to solve basic addition and subtraction problems effectively and to commit the answers to memory will require considerable practice in solving these problems. As described in Chapter 4, the associated practice should be in small doses each day or, at the very least, several times a week. At the beginning of the school year, practice should focus on smaller problems (with sums less than or equal to ten). Large-valued problems should be emphasized in practice once students are skilled at solving the easier problems. Frequent assessment should be provided to determine whether students are mastering new facts and retaining those taught previously. Students have mastered basic facts when they can solve problems involving those facts quickly and accurately. Accurate but slow problem solving indicates that students are still using counting or other procedures to solve simple problems and have not yet committed the basic facts to memory.

Committing the basic addition and subtraction facts to memory is a major objective in the first and second grades. Students who do not commit the basic facts to memory will be at a disadvantage in further work with numbers and arithmetic.

Students have mastered basic facts when they can solve problems involving those facts quickly and accurately.

Understanding the symmetric relationship between sets of simple addition problems, such as 7 +2 and 2 + 7, can be used to reduce the memorization load in learning facts. The teaching of these relationships is to be incorporated into the sequence for teaching students simple addition and during their practice. For example, after students have learned 7 +2, they can be shown that the same answer applies to 2 + 7. Moreover, by placing problems such as 7 + 2 and 2 + 7 in sequence in practice sheets, students will have the opportunity to "discover" and reinforce this relationship as well. Later, they might learn that the combination of 7, 2, and 9 can be used to create subtraction facts and addition facts.

While the standard calls for counting by 1 to 100 in the first grade, counting into the 100s can begin in the latter part of the first grade if students have mastered counting to 100. Counting backward for numbers up to 100 should also be done in the first grade once students have mastered counting forward.

Considerations for Grade-Level Accomplishments in Grade One

The most important mathematical skills and concepts for children in grade one to acquire are described as follows:

- **Reading and writing of numbers.** Many students demonstrate a lack of understanding of place value when they encounter numbers such as 16 and 61. If students are confused by two such similar numbers, teachers should try to determine whether the cause of the confusion is students' failure to understand that numbers are read from left to right or students' inadequate understanding of place value. Instruction should be carefully sequenced to show that 16 is 1 ten and 6 ones, while 61 is 6 tens and 1 one. Students need to know prerequisite skills underlying place value, such as 6 tens equals 60 and its corollary, 60 equals 6 tens, and addition facts in which a single-digit number is added to the tens number, 10 + 3, 10 + 5, 30 + 6. These facts can be taught verbally before students read and write the numbers.

  Learning the number that represents a group of tens is important for understanding place value and reading numbers. Some students are more likely to have difficulty with groups of tens in which the tens number does not say the name of the first digit (e.g., "twenty" is not pronounced "twoty") than with

Grade One

tens numbers in which the name of the first digit is pronounced, sixty, forty, seventy, eighty, ninety. Teachers should provide more practice on the more difficult items.

- **Skip counting.** In addition to enhancing children's number sense, skip counting is important for facilitating the learning of multiplication and division. Counting by tens should be introduced when students can count by ones to about 20 or 30. Counting by tens helps students learn to count by ones to 100. Skip counting is taught just like counting by ones. The teacher models the first part of the sequence; then the students practice the first part. The modeling and practicing continue on new parts of the sequence until students can say the whole sequence. Skip counting requires systematic teaching using a procedure similar to that discussed for counting by ones. Regularly scheduled practice will help students master counting a sequence. Previously introduced sequences should be reviewed as students learn new ones.
- **Teaching of addition and subtraction facts.** Teaching addition and subtraction facts and making assessments should be systematic, as was discussed previously.
- **Understanding of symmetric relationships.** Understanding the symmetric relationship of facts can reduce the number of facts to be memorized in learning.
- **Adding and subtracting of one- and two-digit numbers.** Students can be helped to avoid difficulties with adding one- and two-digit numbers if they are given practice with "lining up" numbers in the problem and adding from right to left. This procedure can be confusing to students because (as previously discussed) we read and write numbers from left to right. Furthermore, in anticipation of subtracting one- and two-digit numbers, students need practice in working from top to bottom.

Grade Two Areas of Emphasis

By the end of grade two, students understand place value and number relationships in addition and subtraction, and they use simple concepts of multiplication. They measure quantities with appropriate units. They classify shapes and see relationships among them by paying attention to their geometric attributes. They collect and analyze data and verify the answers.

Number Sense

1.0	**1.1**	1.2	**1.3**
2.0	**2.1**	**2.2**	2.3
3.0	**3.1**	**3.2**	**3.3**
4.0	**4.1**	**4.2**	**4.3**
5.0	**5.1**	**5.2**	
6.0	6.1		

Algebra and Functions

1.0	**1.1**	1.2	1.3

Measurement and Geometry

1.0	1.1	1.2	**1.3**	1.4	1.5
2.0	**2.1**	**2.2**			

Statistics, Data Analysis, and Probability

1.0	1.1	1.2	1.3	1.4
2.0	2.1	2.2		

Mathematical Reasoning

1.0	1.1	1.2
2.0	2.1	2.2
3.0		

Key Standards

Number Sense

As was the case in grade one, the students' growing mastery of whole numbers is the basic topic in grade two, although fractions and decimals now appear. These Number Sense standards are particularly important:

1.1 Count, read, and write whole numbers to 1,000 and identify the place value for each digit.

1.3 Order and compare whole numbers to 1,000 by using the symbols $<, =, >$.

The following standards are also important in helping students to master whole numbers:

2.1 Understand and use the inverse relationship between addition and subtraction (e.g., an opposite number sentence for $8 + 6 = 14$ is $14 - 6 = 8$) to solve problems and check solutions.

2.2 Find the sum or difference of two whole numbers up to three digits long.

Standard 2.1 gives students a clear application of the relations between different types of operations (addition and subtraction) and can be used to encourage more flexible methods of thinking about and solving problems; for example, a knowledge of addition can facilitate the solving of subtraction problems and vice versa. The problem $144 - 98 = ?$ can be solved by realizing that $144 = 100 + 44 = 98 + 2 + 44 = 98 + 46$.

Standard 2.2 covers the teaching of the addition algorithm for numbers up to three digits. For children at this age, two things should be observed. One is that at the beginning the teaching should be flexible and not insist on the formalism of that algorithm. For example, one can begin the teaching of $23 + 45$ by considering $20 + 3 + 40 + 5 = 20 + 40 + 3 + 5 = 60 + 8 = 68$. This process helps children to become used to the advantage of adding the tens digits and the ones digits separately. A second thing is not to emphasize, at the initial stage, the special skill of "carrying." The key idea of this algorithm is the ability to add the numbers column by column, one digit at a time. In other words the important thing is being able to add digits of the same place (ones digits, tens digits, hundreds digits, and so forth) and still obtain the correct answer at the end. Only after children have learned this concept should the "carrying" skill be taught. The same remark applies to the subtraction algorithm: at the beginning teachers should emphasize that the subtraction of two three-digit numbers can be obtained by performing single-digit subtractions. Thus, $746 - 503$ can be computed from three single-digit subtractions: $7 - 5 = 2$, $4 - 0 = 4$, and $6 - 3 = 3$ so that $746 - 503 = 243$. The teacher can show that this computation is possible because $746 - 503 = 700 + 40 + 6 - 500 - 00 - 3$. The special skill of "trading" needed for a subtraction of $793 - 568$ can be taught only after children thoroughly understand single-digit subtractions. Formal explanations at this grade level are not necessary; friendly persuasion is more appropriate. The mathematical reasoning behind these algorithms is taken up in grade four.

The third Number Sense standard is basic to students' understanding of arithmetic and the ability to solve multiplication and division problems:

3.0 Students model and solve simple problems involving multiplication and division.

Here, fluency with skip counting is helpful. It is important to remind students that multiplication is a shorthand for repeated addition: the meaning of 5×7 is exactly $7 + 7 + 7 + 7 + 7$, no more and no less. This is an opportunity for teachers to impress on students that every symbol and concept in mathematics have a precise, unambiguous meaning.

The discussion of fractions and the goals represented in Number Sense Standards 4.1, 4.2, and 4.3 are also essential features of students' developing arithmetical competencies. Although equivalence of fractions is not explicitly presented in the standards, it is also a good idea to begin the discussion of the topic at this point—students should know, for example, that $\frac{2}{4}$ is the same as $\frac{1}{2}$, a concept that can (and should) be demonstrated with pictures. Finally, as a practical matter and as a basic application of the topics discussed previously, the material in Number Sense Standards 5.1 and 5.2—on modeling and solving problems involving money—is very important. Borrowing money gives a practical context to the concept of subtraction. Special attention should be paid to the need for introducing the symbols $ and ¢ and to the fact that the order of the symbol for dollars is $3, not 3$; but for cents, the order is 31¢, not ¢31.

Although equivalence of fractions is not explicitly presented in the standards, it is a good idea to begin the discussion of the topic at this point.

Algebra and Functions

In the Algebra and Functions strand, the following standard is an *essential* feature of mathematics instruction in grade two:

1.1 Use the commutative and associative rules to simplify mental calculations and to check results.

However, the emphasis here should be on the *use* of these rules to simplify; for example, knowing that $5 + 8 = 13$ saves the labor of also learning that $8 + 5 = 13$. Learning the terminology is not nearly as important. The students should begin to develop an appreciation for the power of unifying rules; but *overemphasizing these topics, particularly the sophisticated concept of the associative rule, is probably worse than not mentioning them at all.*

Measurement and Geometry

Although Standard 1.3 listed below from the Measurement and Geometry strand is important, more emphasis should be given to the topics in Standard 2.0.

1.3 Measure the length of an object to the nearest inch and/or centimeter.

2.0 Students identify and describe the attributes of common figures in the plane and of common objects in space.

Because understanding spatial relations will be more difficult for some students than for others (especially the concepts involving three-dimensional information), teachers should carefully assess how well students understand these shapes and figures and their relationships.

Grade Two

Statistics, Data Analysis, and Probability

Although Standard 1.0 in the Statistics, Data Analysis, and Probability strand is important for grade two, the topics in Standard 2.0 are more important in this grade.

1.0 Students collect numerical data and record, organize, display, and interpret the data on bar graphs and other representations.

2.0 Students demonstrate an understanding of patterns and how patterns grow and describe them in general ways.

But here, as for grade one, it is important that students distinguish between the most likely next term and *the* next term. In statistics students look for likely patterns, but in mathematics students need to know the rule that generates the pattern to determine "the" next term. As an example, given only the sequence 2, 4, 6, 8, 10, students should *not* assert that the next term is 12 but, instead, that the most likely next term is 12. For example, the series might have actually been 2, 4, 6, 8, 10, 14, 16, 18, 20, 22, 26, 28 . . . . The ability to distinguish between what is likely and what is given promotes careful, precise thought.

Elaboration

In the second grade, work on committing the answers to basic addition and subtraction problems to memory should continue for those students who have not mastered them in the first grade. Students' knowledge of facts needs to be assessed at the beginning of the school year. The assessment could be done individually so that the teacher can determine whether the student has committed the facts to memory. Mastery of addition and subtraction facts can also be assessed with simple paper-and-pencil tests. Students should be asked to solve a whole sheet of problems in one or two minutes. As noted earlier, students who have committed the basic facts to memory will quickly and correctly dispose of these simple tasks. If not, they are, most likely, solving the problem by counting in their head (Geary 1994) or using time-consuming counting procedures to generate answers. Additional practice will be necessary for these children.

Students learn the basics of how to "carry" and "borrow" in the second grade. Because carrying and borrowing are difficult for students to master, extended discussion and practice of these skills will likely be necessary (Fuson and Kwon 1992). To carry and borrow correctly, students must understand the base-10 structure of the number system and the concept that carrying and borrowing involve exchanging sets of 10 ones or 10 tens and so forth from one column to the next. It is common for students to incorrectly conceptualize carrying or borrowing; for example, taking a one from the tens column and giving it to the ones column. What has been given, in fact, is one set of 10 units, not one unit from the tens. For example, borrowing in the case of $43 - 7$ can be explained as follows: $43 - 7 = (30 + 13) - 7 = 30 + (13 - 7) = 30 + 6 = 36$, illustrating the associative law of addition in the process. Initially, problems should be limited to

Borrowing illustrates the associative law of addition.

those that require carrying or borrowing across one column (e.g., 17 + 24, 43 – 7), and particular attention should be paid to problems with zero (90 – 34 and 94 – 30) because they are often confusing to students (VanLehn 1990).

Multiplication is introduced in the second grade, and students are to commit to memory the twos, fives, and tens facts. During the initial learning of multiplication, students often confuse addition and multiplication facts, but these confusions should diminish with additional practice. These facts should be taught with the same systematic approach as was discussed for the addition facts in grade one. *The skip counting series for numbers other than 2, 5, and 10 (e.g., 3s, 4s, 9s, 7s, 25s) can be introduced in the second grade to prepare students for learning more multiplication facts in the third grade.* Additionally, the associative and commutative laws can be used to increase the number of multiplication facts the students know. For example, there is no need for students to learn 5×8 if they already know 8×5.

Students in these early grades often have trouble lining numbers up for addition or subtraction. Reminding students to make sure that their numbers are lined up evenly is essential. Students can be taught to use estimation to determine whether their answers are reasonable. However, it is unwise to try to put undue emphasis on estimation by teaching second grade students to answer problems only by making estimates. Instead, they should concentrate on problems that demand an exact answer and use estimation to check whether their answer is reasonable.

The work with fractions should include examples showing fractions that are less than one, fractions that are equal to one, and fractions that are equal to more than one. This range is needed to prevent students from thinking that fractions express only units less than one. To this end, teachers need to make sure that students can freely work with improper fractions and understand that, the name notwithstanding, there is nothing wrong with improper fractions.

Teachers need to make sure that students can freely work with improper fractions.

It has been pointed out that many second grade students have real difficulty with the written form of fractions but much less trouble with their verbal descriptions. Therefore, the verbal descriptions should be emphasized at this level, although students will, of course, eventually need to know the standard written representations of fractions.

Considerations for Grade-Level Accomplishments in Grade Two

The most important mathematical skills and concepts for children in grade two to acquire are described as follows:

- **Counting.** Many students require careful teaching of counting from 100 through 999. Students can learn the counting skills for the entire range through exercises in which the teacher models and provides practice sets consisting of series. First, the teacher models numbers within a particular decade (e.g., 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360). A daily teaching session might include work on several series (e.g., 350 to 360, 140 to

150, 470 to 480). Sets within a decade would be worked on daily until students demonstrate the ability to generalize to new series. During the next stage students would practice on series in which they move from one decade to the next (e.g., 365 to 375, 125 to 135, 715 to 725). Students may have difficulty making the transition from one decade to the next without explicit instruction and adequate practice. When the students demonstrate a general ability to make this transition, the final set of series would be introduced. These sets would include those in which the transition from one one-hundred number to the next occurs: 595 to 605, 195 to 205, 495 to 505.

Grade Two

- **Writing numbers.** If the students are not instructed carefully, some may develop the misconception that the presence of two zeros creates a hundreds number. These students will write three hundred twenty-five as 30025. Teachers should watch for this type of error and correct it immediately. Examples with and without zeros need to be modeled and practiced.

Practice with the terms *more* and *less* and *top* and *bottom* should precede the introduction of problems involving borrowing.

- **Borrowing.** Practice with the terms *more* and *less* and *top* and *bottom* should precede the introduction of problems involving borrowing. These concepts need to be firmly understood if students are to succeed with borrowing problems.
- **Skip counting.** Students should be given opportunities to skip count forward, backward, and starting at any number. Otherwise, students may develop misunderstandings such as it is not possible to count by 2s from an odd number. During the year, students should learn that skip counting by a number starting from zero will also provide a list of multiples for the number. In the process of using skip counting to learn multiples, students may become confused by numbers that appear on several lists. For example, when numbers are counted by threes and fours, the number 12 appears as the fourth number on the "multiples of three" list and as the third number on the "multiples of four" list. To avoid confusing their students, teachers should provide extensive practice with one of these sequences before introducing the next.
- **Counting groups of coins.** This process requires that students be able to say the respective count by series for the value of each coin and be able to answer addition fact questions easily, such as 25 + 5, 30 + 10, in which a nickel or dime is added to a number ending in 5 or 0. Exercises in counting coins should be coordinated with instruction in counting facts so that students have already practiced the skill thoroughly before having to apply it. Counting coins should be reviewed and extended to include quarters along with dimes, nickels, and pennies. A particular fact that some students find difficult to comprehend is adding ten to a two-digit number ending in 5 (e.g., 35 + 10).
- **Aligning columns.** Students may need systematic instruction in rewriting problems written as a column problem; practice in rewriting horizontal equations, such as 304 + 23 =__ or 6 + 345 = ___, in column form; and help in lining numbers up for addition or subtraction. In certain situations they can be taught to use estimation to check whether their answers are reasonable and, if

not, to recheck their work to find their mistakes. As was discussed previously in the subsection on elaboration, it is unwise to try to teach students in grade two to answer problems that request only an estimate as the answer. Students need to become accustomed to obtaining exact answers and using estimation only as an aid to check whether the answer is reasonable.

Students need to become accustomed to obtaining exact answers and using estimation only as an aid to check the answer.

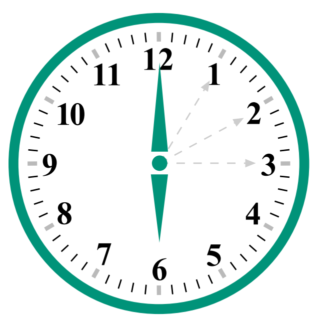
- **Understanding associativity.** Students are expected to know and use the associative attribute of addition and multiplication in the early grades. It is already discussed in the second grade Algebra and Functions, Standard 1.1 (addition), and in the third grade Algebra and Functions, Standard 1.5 (multiplication). Associativity often helps to simplify mental calculations or to verify the correctness of the results and, therefore, its usefulness in those grades.

  However, once subtraction and division have been introduced, the teacher should demonstrate to the students that associativity does not hold for subtraction and division. For example, given the simple subtraction sentence $9 - 4 - 2$, one cannot arbitrarily group the operands because $(9 - 4) - 2$ *is not* equal to $9 - (4 - 2)$. Similarly, in a division sentence such as $18 \div 2 \div 3$, $(18 \div 2) \div 3$ *is not* equal to $18 \div (2 \div 3)$. Such demonstrations, not necessarily in-depth teaching, should occur no later than in the second grade for subtraction and in the fourth grade for division.

- **Reviewing time equivalencies.** Students will need to review time equivalencies (e.g., 1 minute equals 60 seconds, 1 hour equals 60 minutes, 1 day equals 24 hours, 1 week equals 7 days, 1 year equals 12 months). These equivalencies need to be practiced and reviewed so that all students are able to commit them to memory.

- **Understanding money.** In the teaching of decimal notation for money, teachers must ensure that students can read and write amounts such as \$2.05, in which there is a zero in the tenths column, and \$.65, in which there is no dollar amount. By the end of the second grade, students should be able to write ten cents as \$.10 and ten dollars as \$10.00 in decimal notation.

- **Telling time.** Students can be taught a general procedure for telling time. Telling time on an analog clock can begin with teaching students to tell how many minutes after the hour, to the nearest five minutes, are shown on the clock. Students need to be proficient in counting by fives before time telling is introduced. When the students can read the minutes after the hour, reading the minutes before the hour can be introduced. Students should be taught to express the time as minutes after and as minutes before the hour (e.g., 40 minutes after 1 is the same as 20 minutes before 2).



- **Understanding fractions.** Creating a fraction to represent the parts of a whole (e.g., $\frac{2}{3}$ of a pie) is significantly different from dividing a set of items into subgroups and determining the number of items within some subgroups (e.g., $\frac{2}{3}$ of 15). A unit divided into parts can be introduced first, and instruction on that type of fraction should be provided until students can recognize and write fractions to represent fractions of a whole; then the more complex fractions should be introduced. Students can work with diagrams. Computer programs and videos are also available to help with this topic. Students are not expected to solve $\frac{2}{3}$ of 15 numerically in the second grade, because doing so requires them to be able to multiply fractions and convert an improper fraction to a whole number.

Grade Three Areas of Emphasis

By the end of grade three, students deepen their understanding of place value and their understanding of and skill with addition, subtraction, multiplication, and division of whole numbers. Students estimate, measure, and describe objects in space. They use patterns to help solve problems. They represent number relationships and conduct simple probability experiments.

Number Sense

1.0 1.1 1.2 **1.3** 1.4 **1.5**
2.0 **2.1** **2.2** **2.3** **2.4** 2.5 2.6 2.7 2.8
3.0 3.1 **3.2** **3.3** 3.4

Algebra and Functions

1.0 **1.1** 1.2 1.3 1.4 1.5
2.0 **2.1** 2.2

Measurement and Geometry

1.0 1.1 **1.2** **1.3** 1.4
2.0 **2.1** **2.2** **2.3** 2.4 2.5 2.6

Statistics, Data Analysis, and Probability

1.0 1.1 **1.2** **1.3** 1.4

Mathematical Reasoning

1.0 1.1 1.2
2.0 2.1 2.2 2.3 2.4 2.5 2.6
3.0 3.1 3.2 3.3

Key Standards

Number Sense

In the Number Sense strand, Standards 1.3 and 1.5 are especially important:

1.3 Identify the place value for each digit in numbers to 10,000.

1.5 Use expanded notation to represent numbers (e.g., 3,206 = 3,000 + 200 + 6).

For students who show a good conceptual understanding of whole numbers (e.g., place value), the second standard should receive special attention. Here, Standards 2.1, 2.2, 2.3, and 2.4 are especially important:

2.1 Find the sum or difference of two whole numbers between 0 and 10,000.

2.2 Memorize to automaticity the multiplication table for numbers between 1 and 10.

2.3 Use the inverse relationship of multiplication and division to compute and check results.

2.4 Solve simple problems involving multiplication of multidigit numbers by one-digit numbers ($3{,}671 \times 3 = \_\_$).

The foundation that supports Standard 2.1 has been laid in grade two: once students become fluent in adding and subtracting three-digit numbers, increasing the number of digits offers no real difficulty. The new concept in grade three appears in Standard 2.4. Again, the emphasis at the initial stage of teaching the multiplication algorithm should be on the simple cases in which "carrying" plays no role. For example, 234×2 is the same as doubling $200 + 30 + 4$, which is $400 + 60 + 8$, which is 468, which is in turn obtained from 234 by multiplying each digit by 2. The same reasoning applies to 123×3. Once students perceive the possibility that the answer to a multidigit multiplication might be assembled from the answers to simple single-digit problems, the idea of "carrying" can be taught. However, in assembling the answer to such a problem as $234 \times 6 = 200 \times 6 + 30 \times 6 + 4 \times 6$, the fact that the answer can be assembled only from the single-digit multiplications 2×6, 3×6, and 4×6 should be emphasized; this fact makes learning the multiplication table so important.

The relationship between division and multiplication should be emphasized from the beginning.

The relationship between division and multiplication (Standard 2.3) should be emphasized from the beginning. In other words, 39 divided by $3 = 13$ is the same statement as $39 = 13 \times 3$. For children in grade three, a constant reminder of this fact would seem to be necessary.

Two topics in the third standard also deserve special attention:

3.2 Add and subtract simple fractions (e.g., determine that $\frac{1}{8} + \frac{3}{8}$ is the same as $\frac{1}{2}$).

3.3 Solve problems involving addition, subtraction, multiplication, and division of money amounts in decimal notation and multiply and divide money amounts in decimal notation by using whole-number multipliers and divisors.

These are the early introductory elements of arithmetic with fractions and decimals—topics that will build over several years.

Algebra and Functions

In the third grade, the Algebra and Functions strand grows in importance:

1.1 Represent relationships of quantities in the form of mathematical expressions, equations, or inequalities.

Because understanding these concepts can be a very difficult step for students, instruction must be presented carefully, and many examples should be given: 3×12 inches in 3 feet, 4×11 legs in 11 cats, 2×15 wheels in 15 bicycles, 3×15 wheels in 15 tricycles, the number of students in the classroom < 50, the number of days in a year > 300, and so forth.

The next three standards expand on the first and provide examples of what is meant by "represent relationships of . . . ." Teachers must be sure that students are aware of the power of commutativity and associativity in multiplication as a simplifying mechanism and as a means of avoiding overemphasis on pure memorization of the formulas without understanding.

The second standard is also important and likewise must be treated carefully:

2.1 Solve simple problems involving a functional relationship between two quantities (e.g., find the total cost of multiple items given the cost per unit).

Measurement and Geometry

In the first Measurement and Geometry standard, Standards 1.2 and 1.3 should be emphasized:

1.2 Estimate or determine the area and volume of solid figures by covering them with squares or by counting the number of cubes that would fill them.

1.3 Find the perimeter of a polygon with integer sides.

The idea that one cannot talk about area until a square of side 1 has been declared to have unit area and is then used to measure everything else is usually not firmly established in standard textbooks. Analogies should be constantly drawn between length and area. For example, a line segment having a length 3 means that, compared with the segment L that has been declared to be of length 1, it can be covered exactly by 3 nonoverlapping copies of L. Likewise, a rectangle with sides of lengths 3 and 1 has an area equal to 3 because it can be exactly covered by three nonoverlapping copies of the square declared to have length 1.

In the second Measurement and Geometry standard, Standards 2.1, 2.2, and 2.3 are the most important.

2.1 Identify, describe, and classify polygons (including pentagons, hexagons, and octagons).

2.2 Identify attributes of triangles (e.g., two equal sides for the isosceles triangle, three equal sides for the equilateral triangle, right angle for the right triangle).

2.3 Identify attributes of quadrilaterals (e.g., parallel sides for the parallelogram, right angles for the rectangle, equal sides and right angles for the square).

Grade Three

A principal difficulty with geometry at all levels is the need for precise definitions of geometric concepts.

All these standards can be difficult to master if they are presented too generally. A principal difficulty with geometry at all levels is the need for precise definitions of geometric concepts. Even students in grade three need a workable definition of a polygon, a concept that textbooks usually do not supply. A *polygon* may be defined as a finite number of line segments, joined end-to-end, so that together they form the complete boundary of a single planar region. It is strongly recommended that the skills for this grade level be limited to such topics as finding the areas of rectangles with integer sides, right triangles with integer sides, and figures that can be partitioned into such rectangles and right triangles. A few examples in which the sides are not whole numbers should also be provided. Estimation should be used for these examples. Implicit in Standards 2.4 and 2.5 is the introduction of the concept of an angle. But this topic should not be emphasized at this time.

Statistics, Data Analysis, and Probability

The most important standards for Statistics, Data Analysis, and Probability are:

1.2 Record the possible outcomes for a simple event (e.g., tossing a coin) and systematically keep track of the outcomes when the event is repeated many times.

1.3 Summarize and display the results of probability experiments in a clear and organized way (e.g., use a bar graph or a line plot).

Elaboration

In the third grade, work with addition and subtraction problems expands to problems in which regrouping (i.e., carrying and borrowing) is required in more than one column. As noted earlier particularly important and difficult for some students are subtraction problems that include zeros; for example, 302 – 25 and 3002 – 75 (VanLehn 1990). Students need to become skilled in regrouping across columns with zeros because such problems are often used with money applications; for example, *Jerry bought an ice cream for 62 cents and paid for it with a ten-dollar bill. How much change will he receive?*

One way to treat 302 – 25 is again through the use of the associative law of addition: $302 - 25 = (200 + 102) - 25 = 200 + (102 - 25) = 200 + (2 + 100 - 25) = 200 + (2 + 75) = 277$. The first equality is exactly what is meant by "borrowing in the 100s place."

As with addition and subtraction, memorizing the answers to simple multiplication problems requires the systematic introduction and practice of facts. (Refer to the recommendations discussed for addition facts in the first-grade section.) Some division facts can be incorporated into the sequence for learning multiplication facts. As with addition and subtraction, symmetric relationships can be used to cut down on the need for memorization. These related facts can be introduced together (20 divided by 5, 5 times 4).

Grade Three

Multiplication and division problems with multidigit terms are introduced in the third grade (e.g., 36×5). The basic facts used in both types of problems should have already been committed to memory (e.g., students should have already memorized the answer to 6×5, a component of the more complex problem 36×5). Students should already be familiar with the basic structure of these problems because of their understanding of how to add a one-digit to a two-digit number (e.g., $18 + 4$ and $36 + 5$, $12 + 6$). As with addition and subtraction, problems that require carrying (e.g., 36×5) will be more difficult to solve than will the problems that do not require carrying (e.g., 32×4) (Geary 1994).

Memorizing the answers to simple multiplication problems requires the systematic introduction and practice of facts.

The goal is to extend the multiplication of whole numbers up to 10,000 by single-digit numbers (e.g., $9{,}345 \times 2$) so that students gain mastery of the standard right-to-left multiplication algorithm with the multiplier being a one-digit number.

Students are expected to work on long division problems in which they divide a multidigit number by a single digit. A critical component skill for solving these problems is the ability to determine the multiple of the divisor that is just smaller than the number being divided. In $\frac{28}{5}$, the multiple of 5 that is just smaller than 28 is 25. Although the identification of remainders exceeds the level of the third grade standard, students need to become aware of the process for division when there is a remainder. Practice in determining multiples can be coordinated with the practice of multiplication facts. Having basic multiplication facts memorized will greatly facilitate students' ability to solve these division problems.

Rounding is a critical prerequisite for working estimation problems. Noted below is a sequence of exercises that might be followed when introducing rounding. Each exercise can be introduced over several days, followed by continued practice. Practice sets should include examples that review earlier stages and present the current ones, as described in Appendix A, "Sample Instructional Profile."

- Round a 2-digit number to the nearest 10.
- Round a 3-digit number to the nearest 10.
- Round a 3-digit number to the nearest 100.
- Round a 4-digit number to the nearest 1,000.
- Round a 4-digit number to the nearest 100.

The work with fractions in grade three is primarily with diagrams and concrete objects. Students should be able to compare fractions in at least two ways. First, students should be able to order fractions—proper or improper—with like

Grade Three

denominators, initially using diagrams but later realizing that if the denominators are equal, then the order depends only on the numerators. Second, students should be able to order unit fractions, perhaps only with whole-number denominators less than or equal to 6. At this point students are not expected to compare fractions with unlike denominators except for very simple cases, such as $\frac{1}{4}$ and $\frac{3}{8}$ or $\frac{1}{2}$ and $\frac{3}{4}$. Students should compare particular fractions verbally and with the symbols $<, =, >$.

With regard to multiplying and dividing decimals, care should be taken to include exercises in which students have to distinguish between adding and multiplying. Work with money can serve as an introduction to decimals. For example, the following problem is typical of the types of problems that can serve as the introduction of decimal addition:

Josh had $3. He earned $2.50. How much does he have now?

Likewise, the next problem typifies the types of problems that can introduce decimal multiplication:

Josh earned $2.50 an hour. He worked 3 hours. How much did he earn?

The teaching of arithmetic facts can be extended in the third grade to include finding multiples and factors of whole numbers; both are critical to students' understanding of numbers and later to simplifying fractions. Because students need time to develop this skill, it is recommended that they be given considerable instruction on it before they are tested. Only small numbers involving few primes should be used. As a rule, "small" means less than 30, with prime factors limited to only 2, 3, or 5 (e.g., $20 = 2 \times 2 \times 5$, $18 = 3 \times 3 \times 2$).

Considerations for Grade-Level Accomplishments in Grade Three

The most important mathematical skills and concepts for children in grade three to acquire are described as follows:

An assessment of students' knowledge of basic facts needs to be undertaken at the beginning of the school year.

- **Addition and subtraction facts.** Students who enter the third grade without addition and subtraction facts committed to memory are at risk of having difficulty as more complex mathematics is taught. An assessment of students' knowledge of basic facts needs to be undertaken at the beginning of the school year. Systematic daily practice with addition and subtraction facts needs to be provided for students who have not yet learned them.
- **Reading and writing of numbers.** Thousands numbers with zeros in the hundreds or tens place or both (4006, 4060, 4600) can be particularly troublesome for at-risk students. Systematic presentations focusing on reading and writing thousands numbers with one or two zeros need to be provided until students can read and write these more difficult numbers.
- **Rounding off.** Rounding off a thousands number to the nearest ten, hundred, and thousand requires a sophisticated understanding of the rounding-off process. When rounding to a particular unit, students need to learn at which

point to start the rounding process. For example, when rounding off to the nearest hundred, the student needs to look at the current digit in the tens column to determine whether the digit in the hundreds column will remain the same or be increased when rounded off. Practice items need to include a variety of types (e.g., round off 2,375 to the nearest hundred and then to the nearest thousand).

- **Geometry.** While many of these geometric concepts are not difficult in themselves, students typically have difficulty, becoming confused as new concepts and terms are introduced. This problem is solvable through the use of a cumulative manner of introduction in which previously introduced concepts are reviewed as new concepts are introduced.
- **Measurement.** The standards call for students to learn a significant number of measurement equivalencies. These equivalencies should be introduced so that students are not overwhelmed with too much information at one time. Adequate practice and review are to be provided so that students can readily recall all equivalencies.

Grade Four Areas of Emphasis

By the end of grade four, students understand large numbers and addition, subtraction, multiplication, and division of whole numbers. They describe and compare simple fractions and decimals. They understand the properties of, and the relationships between, plane geometric figures. They collect, represent, and analyze data to answer questions.

Number Sense

1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9
2.0 2.1 2.2
3.0 3.1 3.2 3.3 3.4
4.0 4.1 4.2

Algebra and Functions

1.0 1.1 1.2 1.3 1.4 1.5
2.0 2.1 2.2

Measurement and Geometry

1.0 1.1 1.2 1.3 1.4
2.0 2.1 2.2 2.3
3.0 3.1 3.2 3.3 3.4 3.5 3.6 3.7 3.8

Statistics, Data Analysis, and Probability

1.0 1.1 1.2 1.3
2.0 2.1 2.2

Mathematical Reasoning

1.0 1.1 1.2
2.0 2.1 2.2 2.3 2.4 2.5 2.6
3.0 3.1 3.2 3.3

Key Standards

Number Sense

The Number Sense strand for the fourth grade extends students' knowledge of numbers to both bigger numbers (millions) and smaller numbers (two decimal places).

Up to this point students have been asked to learn to round numbers to the nearest tens, hundreds, and thousands, probably without knowing why. It is now finally possible to explain why rounding is much more than a mechanical exercise and that it is in fact an essential skill in the application of mathematics to understanding the world around us. One can use the population figure of the United States for this purpose. According to the latest census (conducted in 2000), there are 281,421,906 people living in this country. The teacher can explain to students that, either in daily conversation or in strategic planning, using the rounded-off figure of 280 million instead of the precise figure of 281,421,906 would be more sensible, because a project of this size has built-in errors and correctly counting all the people in transit, reaching all homeless people, and obtaining total participation are impossible. Therefore, rounding to the nearest ten million in this case becomes a matter of necessity in discarding unreliable and nonessential information.

Standard 1.5 brings out two facts about fractions that are fundamental for students' understanding of this topic: different interpretations of a fraction and the equivalence of fractions. These facts will be discussed one at a time.

The fact that a fraction such as $\frac{3}{5}$ is not only 3 parts of a whole when the whole (the unit) is divided into 5 equal parts but also one part of 3 when 3 is divided into 5 equal parts is so basic that one often uses it without being aware of doing so. For example, if someone is asked in a daily conversation how long one of the pieces of a 3-foot rod is when it is cut into 5 pieces of equal length, he or she would say without thinking that it is $\frac{3}{5}$ of a foot. In so doing that person is using the second (division) interpretation of $\frac{3}{5}$. On the other hand, it is important to remember that, *according to the part-whole definition of a fraction,* $\frac{3}{5}$ of a foot is the length of 3 of the pieces when a 1-foot rod is divided into 5 pieces of equal length. *Students need an explanation of why these two lengths are equal.* One way to explain is to divide each foot of the 3-foot rod into five equal sections, as shown in figure 1.

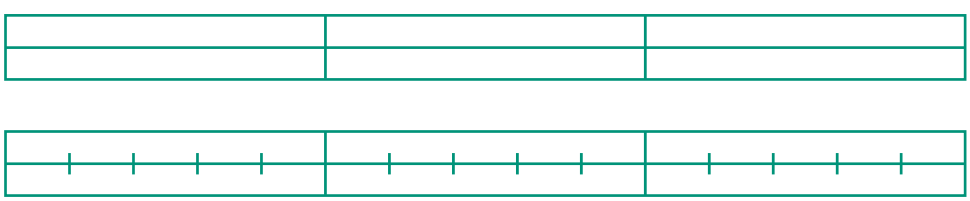


Figure 1

Each section is the result of dividing 1 foot into 5 equal parts, and so by the part-whole definition of a fraction, the length of three such sections joined together, as shown in figure 2, is $\frac{3}{5}$ of a foot.

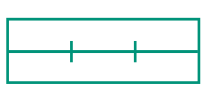


Figure 2

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But the 15 (= 3×5) sections of the 3-foot rod can be grouped to divide the rod into five equal lengths, as shown in figure 3, and it is evident that $\frac{3}{5}$ of a foot is identical to the length of one of the pieces when a 3-foot rod is divided into 5 equal lengths.

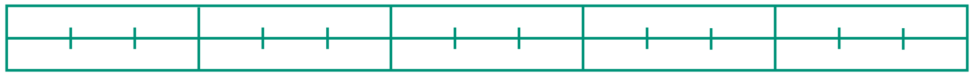

Figure 3

Therefore the part-whole and division definitions of a fraction coincide.

This explanation continues to be valid when the fraction $\frac{3}{5}$ is replaced by any other fraction $\frac{a}{b}$.

The concept of the equivalence of fractions lies at the core of almost every mathematical consideration related to fractions.

The concept of the equivalence of fractions lies at the core of almost every mathematical consideration related to fractions. Students should be given every opportunity to understand why $\frac{2}{5} = \frac{14}{35}$, why $\frac{5}{4} = \frac{40}{32}$, or why $\frac{a}{b} = \frac{na}{nb}$ for any whole number *a, b, n* (it will always be understood that $b \neq 0$ and $n \neq 0$). One can use a picture to explain why $\frac{2}{5} = \frac{14}{35}$, provided that the context of the picture is carefully laid out. Let the unit 1 be fixed as the *area* of the unit square, as shown in figure 4.

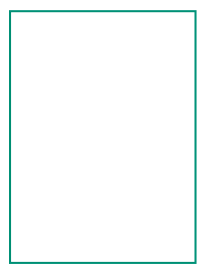

Figure 4

The fraction $\frac{2}{5}$ is then 2 parts of the unit square when it is divided into 5 parts of equal area. The equidivision is done vertically, as shown in figure 5.

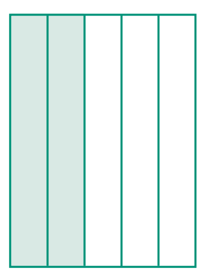

Figure 5

Since each vertical strip represents $\frac{1}{5}$, the shaded region represents $\frac{2}{5}$. The fraction $\frac{14}{35}$ is, on the other hand, 14 parts of the unit square when it is divided into 35 parts of equal area. The desired equidivision into 35 parts can be achieved by adding 7 equally spaced horizontal divisions of the unit square to the preceding vertical division, as shown in figure 6.

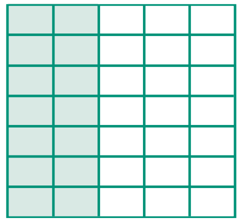

Figure 6

The unit square is now divided into 35 small rectangles of the same size, so that each small rectangle is $\frac{1}{35}$. Since there are 14 of these small rectangles in the shaded region, it therefore represents not only $\frac{2}{5}$ but also $\frac{14}{35}$.

The preceding reasoning is general, but for fourth graders mentioning $\frac{a}{b} = \frac{na}{nb}$ in passing may be enough. What needs special emphasis, however, is the immediate consequence of the equivalence of $\frac{a}{b}$ and $\frac{na}{nb}$, namely, that *any two fractions can be written as two fractions with the same denominator.* Thus if $\frac{a}{b}$ and $\frac{c}{d}$ are two given fractions, they can be rewritten as $\frac{ad}{bd}$ and $\frac{bc}{bd}$, which have the same denominator *bd*. This fact has enormous implications when students come to the addition of fractions.

The consideration of why a fraction has a division interpretation, as explained previously, also sheds light on the teaching of Standard 1.7. To represent the fraction $\frac{3}{5}$ as a decimal, for example, we divide the given unit into 10 equal parts. This concept is best represented on the number line as 9 equidistant markings of the line segment from 0 to 1. By taking the second, fourth, sixth, and eighth markings, we obtain a division of the unit into 5 equal parts. Since the fraction $\frac{3}{5}$ is 3 of these parts, it is the sixth marking. But the 10 markings represent 0.1, 0.2, . . . 0.9; therefore, the sixth marking is 0.6. This process shows that $\frac{3}{5}$ is 0.6.

The next standards are basic and new:

1.8 Use concepts of negative numbers (e.g., on a number line, in counting, in temperature, in "owing").

1.9 Identify on the number line the relative position of positive fractions, positive mixed numbers, and positive decimals to two decimal places.

These standards can be difficult for students to learn if the required background material—ordering of whole numbers and comparison of fractions and decimals—is not presented carefully. The importance of these standards requires that close attention be paid to assessment. Standard 1.9 is about "simple" decimals, that is, decimals up to two decimal places. It is time to note that the addition and subtraction of decimals up to two decimal places can be completely modeled through the use of money and can therefore be done informally. To prepare to study, in grade five, the arithmetic operations of (finite or terminating) decimals of any number of decimal digits, students need to know that, formally, a *finite decimal* is a fraction whose denominator is a power of 10. This awareness is

Students need to know that, formally, a *finite decimal* is a fraction whose denominator is a power of 10.

Grade Four

important in the teaching of decimals in grade four. (To develop this awareness, the teacher can describe decimals such as 1.03 verbally as one and three-hundredths, not as one point oh three).

The third topic in the Number Sense strand is also especially important. Standard 3.0 and its four substandards involve the use of the standard algorithms for addition, subtraction, and multiplication of multidigit numbers and the standard algorithm for division of a multidigit number by a one-digit number. As with simple arithmetic, mastery of these skills will require extensive practice over several grade levels, as described in Chapter 4, "Instructional Strategies." The emphasis in Standard 3.1 is, however, on a formal (mathematical) understanding of the addition and subtraction algorithms for whole numbers. Students need to see the prominent role that the commutative law and, especially, the associative law of addition play in the explanation of these algorithms. The students' prior familiarity with the skill component of these algorithms is essential here because if students do not clearly understand the mechanics of these algorithms, they will be preoccupied with the mechanics and not be free to appreciate the reasoning behind the mechanics.

Standard 3.2 is about the reasoning that supports the multiplication and division algorithms at least in simple situations (two-digit multipliers and one-digit divisors). Introducing this standard is a bit awkward here because the key fact is the distributive law, which is not mentioned until grade five (Algebra and Functions, Standard 1.3). However, if the concept is presented carefully and patiently, students can learn the distributive law. For the division algorithm there is a new element, namely, division-with-remainder: if a and b are whole numbers, then there are always whole numbers q and r so that $a = qb + r$, where r is a whole number strictly smaller than the divisor b. The division algorithm can then be explained as an iterated, or repeated, application of this division-with-remainder.

Students who understand prime numbers will find it easier to understand the equivalence of fractions and to multiply and divide fractions in grades five, six, and seven.

Standard 4.0, "Students know how to factor small whole numbers," is needed for the discussion of the equivalence of fractions. Standard 4.2 contains the requirement that students understand what a prime number is. The concept of primality is important yet often difficult for students to understand fully. Students should also know the prime numbers up to 50. For these reasons the preparation for the discussion of prime numbers should begin no later than the third grade. Students who understand prime numbers will find it easier to understand the equivalence of fractions and to multiply and divide fractions in grades five, six, and seven.

Algebra and Functions

In the fourth grade the Algebra and Functions strand continues to grow in importance. All five of the subtopics under the first standard are important. But the degree to which students need to understand these strands differs. The following standards *do not need undue emphasis*:

1.2 Interpret and evaluate mathematical expressions that now use parentheses.

1.3 Use parentheses to indicate which operation to perform first when writing expressions containing more than two terms and different operations.

These standards involve nothing more than notation. The real skill is learning how to write expressions unambiguously so that others can understand them. However, it would be appropriate at this point to explain carefully to students why the associative and commutative laws are significant and why arbitrary sums or products, such as $115 + 6 + (-6) + 4792$ or $113 \times 212 \times 31 \times 11$, do not have to be ordered in any particular way, nor do they have to be calculated in any particular order.

Standards 1.4 and 1.5, which relate to functional relationships, are much more important theoretically. In particular, students should understand Standard 1.5 because it takes the mystery out of the topic.

1.5 Understand that an equation such as $y = 3x + 5$ is a prescription for determining a second number when a first number is given.

One way to understand an equation such as $y = 3x + 5$ is to work through many pairs of numbers (x, y) to see if they satisfy this equation. For example, (1, 8) and (0, 5) do, but (–1, 3) and (2, 10) do not.

The second algebra standard is, however, basic:

2.0 Students know how to manipulate equations.

This standard and the two basic rules that follow, if understood now, will clarify much of what happens in mathematics and other subjects from the fifth grade through high school.

2.1 Know and understand that equals added to equals are equal.

$2 + 1 = 3$, and $7 - 2 = 5$; therefore, $2 + 1 + 5 = 3 + 7 - 2$.

2.2 Know and understand that equals multiplied by equals are equal.

$2 + 1 = 3$, and $4 \times 5 = 20$; therefore, $(2 + 1) \times 20 = 3 \times (4 \times 5)$.

However, if these concepts are not clear, difficulties in later grades are virtually guaranteed. Therefore, careful assessment of students' understanding of these principles should be done here.

Measurement and Geometry

The Measurement and Geometry strand for the fourth grade contains a few key standards that students will need to understand completely. The first standard (1.0) relates to perimeter and area. The students need to understand that the area of a rectangle is obtained by multiplying length by width and that the perimeter is given by a linear measurement. The intent of most of this standard is that students know the reasons behind the formulas for the perimeter and area of a rectangle and that they can see how these formulas work when the perimeter and area vary as the rectangles vary.

A more basic standard is the second one:

2.0 Students use two-dimensional coordinate grids to represent points and graph lines and simple figures.

Although the material in this standard is basic and is not presented in depth, this concept must be presented carefully. Again, students who are confused at this point will very likely have serious difficulties in the later grades—not just in mathematics, but in the sciences and other areas as well. Therefore, careful assessment is necessary. Special attention should be given to the need for students to understand the graphs of the equations $x = c$ and $y = c$ for a constant c. These graphs are commonly called vertical and horizontal lines, respectively. Students need to be able to locate some points on these graphs strictly according to the definition of the graph of an equation as the set of all points (x, y) whose coordinates satisfy the given equation. Unless this process is painstakingly done, these graphs will continue to be nothing but magic throughout the rest of students' schooling.

In connection with Standard 3.0, teachers should introduce the symbol $\perp$ for perpendicularity. Incidentally, this is the time to introduce the abbreviated notation ab in place of the cumbersome $a \times b$.

Elaboration

Knowledge of multiplication and division facts should be reassessed at the beginning of the school year, and systematic instruction and practice should be provided to enable students to reach high degrees of automaticity in recalling these facts. This process is described for addition in grade two (see "Elaboration").

Students need to understand that zeros in different positions represent different place values—tens, hundreds, thousands, and so forth.

Reading and writing thousands and millions numbers with one or more zeros in the middle can be particularly troublesome for students (Seron and Fayol 1994). Therefore, assessment and teaching should be thorough so that students are able to read and write difficult numbers, such as 300,200 and 320,000. Students need to understand that zeros in different positions represent different place values—tens, hundreds, thousands, and so forth—and they need practice in working with these types of numbers (e.g., determining which is larger, 320,000 or 300,200, and translating a verbal label, "one million two hundred thousand," into the Arabic numeral representation, 1,200,000).

To be able to apply mathematics in the real world, to understand the way in which numbers distribute on the number line, and ultimately to study more advanced topics in mathematics, students need to understand the concept of "closeness" for numbers. It is probably not wise to push too hard on the notion of "close enough" while students are still struggling with the abstract idea of a number itself. However, by now they should be ready for this next step. A discussion of rounding should emphasize that one rounds off only if the result of rounding is "close enough."

Students need to understand fraction equivalencies related to the ordering and comparison of decimals. Students must understand, for instance, that $\frac{2}{10} = \frac{20}{100}$, then equate those fractions to decimals.

The teaching of the conversion of proper and improper fractions to decimals should be structured so that students see relationships (e.g., the fraction $\frac{7}{4}$ can be converted to $\frac{4}{4} + \frac{3}{4}$, which in turn equals 1 and $\frac{3}{4}$). The fourth grade standards do not require any arithmetic with fractions; however, practice with addition and subtraction of fractions (converting to like denominators) must be continued in this grade because these concepts are important in the fifth grade. Students can also be introduced to the concept of unlike denominators in preparation for the following year. Building students' skills in finding equivalent fractions is also important at this grade level.

The standards require that students know the definition of prime numbers and know that many whole numbers decompose into products of smaller numbers in different ways. Using the number 150 as an example, they should realize that $150 = 5 \times 30$ and $30 = 5 \times 6$; therefore, $150 = 5 \times 5 \times 6$, which can be decomposed to $5 \times 5 \times 3 \times 2$. Students will be using these factoring skills extensively in the later grades. Even though determining the prime factors of all numbers through 50 is a fifth grade standard, practice on finding prime factors can begin in the fourth grade. Students should be given extensive practice over an extended period of time with finding prime factors so that they can develop automaticity in the factoring process (see Chapter 4, "Instructional Strategies"). By the end of the fifth grade, students should be able to determine with relative ease whether any of the prime numbers 2, 3, 5, 7, or 11 are factors of a number less than 200.

Multiplication and division problems with multidigit numbers are expanded. Division problems with a zero in the quotient (e.g., $\frac{4233}{6} = 705.5$) can be particularly difficult for students to understand and require systematic instruction.

The Number Sense Standards 3.1 and 3.2 call for "understanding of the standard algorithm" (see the glossary). To present this concept, the teacher sketches the reasons why the algorithm works and carefully shows the students how to use it. (Any such explanation of the multiplication and division algorithms would help students to deepen their understanding and appreciation of the distributive law.) The students are not expected to reproduce this discussion in any detail, but they are expected to have a general idea of why the algorithm works and be able to expand it in detail for small numbers.

Any such explanation of the multiplication and division algorithms would help students to deepen their understanding and appreciation of the distributive law.

As the students grow older, this experience should lead to increased confidence in understanding these *and similar* algorithms, knowledge of how to construct them in other situations, and the importance of verifying their correctness before relying on them. For example, the process of writing any kind of program for a computer begins with creating algorithms for automating a task and then implementing them on the machine. Without hands-on experience like that described previously, students will be ill-equipped to construct correct programs.

Grade Four

Considerations for Grade-Level Accomplishments in Grade Four

The most important mathematical skills and concepts for students in grade four to acquire are described as follows:

Students' knowledge of basic facts needs to be assessed at the beginning of the school year.

- **Multiplication and division facts.** Students who enter the fourth grade without multiplication facts committed to memory are at risk of having difficulty as more complex mathematics is taught. Students' knowledge of basic facts needs to be assessed at the beginning of the school year. Systematic daily practice with multiplication and division facts needs to be provided for students who have not yet learned them.
- **Addition and subtraction.** Mentally adding a two-digit number and a one-digit number is a component skill for working multiplication problems that was targeted in the second grade. Students have to add the carried number to the product of two factors (e.g., 34×3). Students should be assessed on the ability to add numbers mentally (e.g., $36 + 7$) at the beginning of the school year, and systematic practice should be provided for students not able to work the addition problems mentally.
- **Reading and writing numbers.** Reading and writing numbers in the thousands and millions with one or more zeros in the middle can be particularly troublesome for students. Assessment at the beginning of the fourth grade should test students on reading and writing the more difficult thousand numbers, such as 4,002 and 4,020. When teaching students to read 5- and 6-digit numbers, teachers should be thorough so that students can read, write, and distinguish difficult numbers, such as 300,200 and 320,000.
- **Fractions equal to one.** Understanding fractions equal to one (e.g., $\frac{8}{8}$ or $\frac{4}{4}$) is important for understanding the procedure for working with equivalent fractions. Students should have an in-depth understanding of how to construct a fraction that equals one to suit the needs of the problem; for example, should a fraction be $\frac{32}{32}$ or $\frac{17}{17}$? When the class is working on equivalent fraction problems, the teacher should prompt the students on how to find the equivalent fraction or the missing number in the equivalent fraction. The students find the fraction of one that they can use to multiply or divide by to determine the equivalent fraction. (This material is discussed in depth in Appendix A, "Sample Instructional Profile.")

- **Multiplication and division problems.** Multiplication problems in which either factor has a zero are likely to cause difficulties. Teachers should provide extra practice on problems such as 20×315 and 24×308. Division problems with a zero in the answer may be difficult for students (e.g., $\frac{1521}{3}$ and $\frac{5115}{5}$). Students will need prompting on how to determine whether they have completed the problem of placing enough digits in the answer. (Students who consistently find problems with zeros in the answer difficult to solve may also have difficulties with the concept of place value. Help should be provided to remedy this situation quickly.)
- **Order of operations.** In the fourth grade students start to handle problems that freely mix the four arithmetic operators, and the order of operation needs to be addressed explicitly. Students already need to know the convention of order of operations, the precedence of multiplication and division over addition and subtraction, and the implied left-to-right order of evaluation. Parentheses introduce a new way to modify that convention, and Algebra and Functions (AF) Standard 1.2 explicitly addresses this topic.

  The fourth grade is also the time to expose the students to the *convenience* of this convention. Students have already been taught that an equation is a prescription to determine a second number when a first number is given (AF Standard 1.5) in problems and in number sentences, and the clarity of $5x + 3$ over $(5x) + 3$ can be easily demonstrated. This is also the proper time to start moving students away from using the explicit notation of the multiplication symbol, comparing such expressions as $5 \times A + 3$ or $5 \cdot A + 3$ with $5A + 3$. By grade six the topic of order of operations should be mastered.

  A comparison should be made between the associativity of addition and multiplication versus the nonassociativity of subtraction and division. A demonstration should be given of how replacement of subtraction by the equivalent addition of negative numbers, or multiplication with a reciprocal instead of division, solves the associativity problem. In other words the nonassociativity of the sentence

  $$(9 - 4) - 2 \neq 9 - (4 - 2)$$

  should be compared with the restored associativity when subtraction is replaced with addition of the negative value:

  $$[9 + (-4)] + (-2) = 9 + [(-4) + (-2)]$$

  In a similar fashion, although there is no associativity with division,

  $$(18 \div 2) \div 3 \neq 18 \div (2 \div 3),$$

  when the division is replaced with the multiplication by a reciprocal, the associativity returns:

  $$\left(18 \cdot \frac{1}{2}\right) \cdot \frac{1}{3} = 18 \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right)$$

Now, finally, the student can be exposed to the complete reasoning behind the convention of order of operations. The awkward replacement by the inverse operations, or the need for parentheses, can be much reduced by the application of left-to-right evaluation and the precedence of operators. Is it clearer to write $3a^2 - 5a + 3$ instead of $(3 \cdot (a^2)) - (5a) + 3$?

However, students should remember that mathematical writing also serves to *communicate*. Therefore, if an expression is complex and can easily be misinterpreted, a pair of parentheses may be inserted, even if they are not strictly required. Students should be encouraged to write $8 - ((12 \div 4) \div 2) \cdot 3 + 3$ instead of $8 - 12 \div 4 \div 2 \cdot 3 + 3$ because in the first expression, it is less tempting to incorrectly divide 4 by 2 or to incorrectly multiply 2 by 3. The use of a horizontal fraction line for division, such as $\frac{a}{b}$ instead of the division symbol $a \div b$, and the liberal use of spaces, should also be encouraged to enhance readability and reduce errors. Surely $8 - \frac{12 \cdot 3}{4 \cdot 2} + 3$ is even clearer and less error prone than any one of the previous two forms of the same expression.

Grade Five Areas of Emphasis

By the end of grade five, students increase their facility with the four basic arithmetic operations applied to fractions and decimals and learn to add and subtract positive and negative numbers. They know and use common measuring units to determine length and area and know and use formulas to determine the volume of simple geometric figures. Students know the concept of angle measurement and use a protractor and compass to solve problems. They use grids, tables, graphs, and charts to record and analyze data.

Number Sense

1.0 1.1 1.2 1.3 1.4 1.5
2.0 2.1 2.2 2.3 2.4 2.5

Algebra and Functions

1.0 1.1 1.2 1.3 1.4 1.5

Measurement and Geometry

1.0 1.1 1.2 1.3 1.4
2.0 2.1 2.2 2.3

Statistics, Data Analysis, and Probability

1.0 1.1 1.2 1.3 1.4 1.5

Mathematical Reasoning

1.0 1.1 1.2
2.0 2.1 2.2 2.3 2.4 2.5 2.6
3.0 3.1 3.2 3.3

Key Standards

A significant development in students' mathematics education occurs in grade five. From grades five through seven, a three-year sequence begins that provides the mathematical foundation of rational numbers. Fractions and decimals have been taught piecemeal up to this point. For example, only decimals with two decimal places are discussed in the fourth grade, and only fractions with the same denominator (or if one denominator is a multiple of the other) are added or subtracted up to grade four. Now both fractions and decimals will be systematically discussed during the next three years. The demand on students' ability to reason goes up ever so slightly at this point, and the teaching of mathematics must correspondingly reflect this increased demand.

Grade Five

By the time students have finished the fourth grade, they should have a basic understanding of whole numbers and some understanding of fractions and decimals. Students at this grade level are expected to have mastered multiplication and division of whole numbers. They should also have had some exposure to negative numbers. These skills will be enhanced in the fifth grade. An important standard focused on enhancing these skills is Number Sense Standard 1.2.

Number Sense

1.2 Interpret percents as a part of a hundred; find decimal and percent equivalents for common fractions and explain why they represent the same value; compute a given percent of a whole number.

A fraction $\frac{c}{d}$ is both "*c* parts of a whole consisting of *d* equal parts" and "the quotient of the number *c* divided by the number *d*."

The fact that a fraction $\frac{c}{d}$ is both "*c* parts of a whole consisting of *d* equal parts" and "the quotient of the number *c* divided by the number *d*" was first mentioned in Number Sense Standard 1.5 of grade four. As discussed earlier in the section on grade four, this fact must be *carefully explained* rather than decreed by fiat, as is the practice in most school textbooks. The importance of providing logical explanations for all aspects of the teaching of fractions cannot be overstated because the students' fear of fractions and the mistakes related to them appear to underlie the failure of mathematics education. Once $\frac{c}{d}$ is clearly understood to be the division of *c* by *d*, then the conversion of fractions to decimals can be explained logically.

Students will also continue to learn about the relative positions of numbers on the number line, above all, those of negative whole numbers. Negative whole numbers are especially important because, for the first time, they play a major part in core number-sense expectations. Standard 1.5 is important in this regard.

1.5 Identify and represent on a number line decimals, fractions, mixed numbers, and positive and negative integers.

The correct placement of positive fractions on the number line implies that students will need to order and compare fractions. Identifying numbers as points on the real line is an important step in relating students' concepts of arithmetic to geometry. This fusion of arithmetic and geometry, which is ubiquitous in mathematics, adds a new dimension to students' understanding of numbers.

Inasmuch as the study of mixed numbers is one of the things that terrorize elementary school students, the teacher must approach Standard 1.5 carefully. First, students should not be made to think of "proper" and "improper" fractions as distinct objects; they should be helped to understand that these types of fractions are nothing more than different examples of the same concept—namely, a fraction. Identifying fractions as points on the number line (so that one point is no different from any other point) would go a long way toward eliminating most of this misconception. With that understood, the teacher can now mention that for improper fractions, there is an *alternate representation*. For example, on the number line $\frac{5}{4}$ is beyond 1 by the amount of $\frac{1}{4}$, so $1\frac{1}{4}$ is a reasonable alternate notation. Similarly, $\frac{11}{3}$ is $\frac{2}{3}$ beyond 3 on the number line, so $3\frac{2}{3}$ is also a reasonable alternate notation. When a fraction such as $\frac{5}{4}$ or $\frac{11}{3}$ is written as $1\frac{1}{4}$ or $3\frac{2}{3}$, it is said to be a mixed number. In general, fifth graders should be ready for the general explanation of how to write an improper fraction as a mixed number through the use of division-with-remainder. For example, if $\frac{a}{b}$ is an improper fraction, it can be rewritten as a mixed number in the following way: The division of the whole number a by the whole number b is expressed as $a = qb + r$, where q is the quotient and the remainder r is the whole number strictly less than b. Then the fraction $\frac{a}{b}$ is, *by definition*, written as the mixed number $q\frac{r}{b}$. Notice that $\frac{r}{b}$ is a proper fraction. The important point to emphasize is that a mixed number is just a clearly prescribed way of rewriting a fraction, and no fear needs to be associated with it.

But the most important aspect of students' work with negative numbers is to learn the rules for doing the basic operations of arithmetic with them, as represented in the following standard:

2.1 Add, subtract, multiply, and divide with decimals; add with negative integers; subtract positive integers from negative integers; and verify the reasonableness of the results.

In the fifth grade students learn how to add negative numbers and how to subtract positive numbers from negative numbers. At this point students should find it profitable to interpret these concepts geometrically. Adding a positive number b shifts the point on the number line to the right by b units, and adding a negative number $-b$ shifts the point on the number line to the left by b units, and so forth. Multiplication and division of negative numbers should not be taken up in the fifth grade because division by negative numbers leads to negative fractions, which have not yet been introduced. Although Standard 2.1 is listed before Standards 2.3 and 2.4 on the addition and multiplication of fractions, the teaching of decimals must rest on the concept of fractions and their arithmetic operations. A *finite decimal* is formally defined as a fraction whose denominator is a power of 10. Without this precise definition, it is difficult to explain why the addition and subtraction of decimals are reduced to the addition and subtraction of whole numbers so that the algorithms of whole numbers can be applied. More to the point, without this precise definition, it would be essentially impossible to explain the rule regarding the decimal point in the multiplication and division of decimals. For example, 2.4×0.37 can be computed by $24 \times 37 = 888$, and since there are three decimal places in both numbers altogether, the usual rule says $2.4 \times 0.37 = 0.888$. The reason, based on the precise definition of a decimal, is that, by definition, $2.4 = \frac{24}{10}$ and $0.37 = \frac{37}{100}$ so that

$$2.4 \times 0.37 = \left(\frac{24}{10}\right) \times \left(\frac{37}{100}\right) = \frac{(24 \times 37)}{1000} = \frac{888}{1000} = 0.888.$$

In the fifth grade students learn how to add negative numbers and how to subtract positive numbers from negative numbers.

In many textbooks the arithmetic operations of decimals precedes the discussion of fractions, and in general a definition of decimals is not provided. This organization of content creates difficulty for the classroom teacher.

Grade Five

The introduction of the general division algorithm is also important, but it can be complicated and consequently difficult for many students to master. In particular, the skills needed to find the largest product of the divisor with an integer between 0 and 9 that is less than the remainder are likely to be demanding for fifth grade students. Students should become comfortable with the algorithm in carefully selected cases in which the numbers needed at each step are clear. Putting such a problem in context may help. For instance, the students might imagine dividing 153 by 25 as packing 153 students into a fleet of buses for a field trip, with each bus carrying a maximum of 25 passengers. Drawing pictures to help with the reasoning, if necessary, can help students to see that it takes six buses with three students left over; those three students get to enjoy being in the seventh bus with room to spare. But it seems both unnecessary and unwise at this stage to extend the concepts beyond what is presented here. The important standard for students to achieve is:

2.2 Demonstrate proficiency with division, including division with positive decimals and long division with multidigit divisors.

The most essential number-sense skills that students should learn in the fifth grade are the addition and subtraction of fractions.

The most essential number-sense skills that students should learn in the fifth grade are the addition and subtraction of fractions (Standard 2.3) and, to a lesser degree, the multiplication and division of fractions (Standards 2.4 and 2.5). At this point of students' mathematics education, they need to recognize fractions as numbers that are similar to whole numbers and can therefore be added, multiplied, and so forth. In other words fractions are a special collection of points on the number line that include the whole numbers. To add $\frac{a}{b}+\frac{c}{d}$, for example, students can look to the addition of whole numbers for a model. Since $3 + 8$ is the length of the combined segments when a segment of length 3 is concatenated with, or linked to, a segment of length 8, likewise $\frac{a}{b}+\frac{c}{d}$ can be defined as the length of the combined segments when a segment of length $\frac{a}{b}$ is linked to a segment of length $\frac{c}{d}$. The computation of this combined length is complicated by the fact that b may be different than d. But the concept of equivalent fractions shows how any two fractions can be made to have the same denominator, namely, $\frac{a}{b}=\frac{ad}{bd}$ and $\frac{c}{d}=\frac{cb}{bd}$. Therefore, if $\frac{1}{bd}$ is the basic unit, then $\frac{a}{b}$ is ad copies of such a unit, and $\frac{c}{d}$ is bc copies of such a unit. Combining them, therefore, shows that $\frac{a}{b}+\frac{c}{d}$ is $ad + bc$ copies of such a unit $\frac{1}{bd}$; that is, $\frac{a}{b}+\frac{c}{d}=\frac{(ad+bc)}{bd}$.

This example is a simple way to obtain a formula for adding fractions. But this formula is not the definition of adding fractions, which is modeled after the addition of whole numbers. The addition of fractions in terms of the least common multiple of the denominators has struck fear in students for many generations and should never have been used for the definition of adding fractions. Finding the least common multiple is a special skill that should be learned, but it is not how students should think of the addition of fractions.

Once students have mastered these basic skills with fractions, problems involving concrete applications can be used to provide practice and to promote students' technical fluency with fractions.

Two main skills are involved in reducing fractions: factoring whole numbers in order to put fractions into reduced forms and understanding the basic arithmetic skills involved in this factoring. The two associated standards that should be emphasized are:

1.4 Determine the prime factors of all numbers through 50 and write the numbers as the product of their prime factors by using exponents to show multiples of a factor (e.g., $24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$).

2.3 Solve simple problems, including ones arising in concrete situations involving the addition and subtraction of fractions and mixed numbers (like and unlike denominators of 20 or less), and express answers in the simplest form.

The instructional profile with fractions, which appears later in Appendix A, gives many ideas on how to approach this topic. Students may profit from the use of the Sieve of Eratosthenes (see the glossary) in connection with Standard 1.4.

Standard 2.4 introduces the multiplication and division of fractions. This topic will be taken up in earnest in grade six, but it is important at this point to remind students of the meaning of division among whole numbers as an *alternate way of writing multiplication*. In other words if $4 \times 7 = 28$, then, *by definition*, $28 \div 7 = 4$, or in general, if $a \times b = c$, then $c \div b = a$. Teachers can use drills or manipulatives to help students to understand the idea of "division as a different expression of multiplication." Once students have learned this concept, they will be ready for the corresponding situation with fractions; that is, if a, b, and c are fractions, then again by definition, $a \times b = c$ means the same as $c \div b = a$. Using simple fractions, such as $b = \frac{1}{2}$ or $\frac{1}{3}$ and $c = 6$ or 24, and by drawing pictures if necessary, one can easily illustrate why $12 \times \frac{1}{2} = 6$ is the same as there are 12 copies of $\frac{1}{2}$ in 6 (i.e., $6 \div \frac{1}{2} = 12$) or why $24 \times \frac{1}{3} = 8$ is the same as there are 24 copies of $\frac{1}{3}$ in 8 (i.e., $8 \div \frac{1}{3} = 24$).

Drills or manipulatives can help students to understand idea of "division as a different expression of multiplication."

Algebra and Functions

The Algebra and Functions strand for grade five presents one of the key steps in abstraction and one of the defining steps in moving from simply learning arithmetic to learning mathematics: the replacement of numbers by variables.

1.2 Use a letter to represent an unknown number; write and evaluate simple algebraic expressions in one variable by substitution.

The importance of this step, which requires *reasoning rather than simple manipulative facility,* mandates particular care in presenting the material. The basic idea that, for example, $3x + 5$ is a shorthand for an infinite number of sums, $3(1) + 5$, $3(2.4) + 5$, $3(11) + 5$, and so forth, must be thoroughly presented and understood by students; and they must practice solving simple algebraic expressions. But it is probably a mistake to push too hard here—teachers should not overdrill. Instead, they should check for students' understanding of concepts, perhaps providing students with some simple puzzle problems to give them practice in writing an equation for an unknown from data in a word problem.

Grade Five

Again, in the Algebra and Functions strand, the following two standards are basic:

1.4 Identify and graph ordered pairs in the four quadrants of the coordinate plane.

1.5 Solve problems involving linear functions with integer values; write the equation; and graph the resulting ordered pairs of integers on a grid.

Measurement and Geometry

In Measurement and Geometry these three standards should be emphasized:

1.1 Derive and use the formula for the area of a triangle and of a parallelogram by comparing each with the formula for the area of a rectangle (i.e., two of the same triangles make a parallelogram with twice the area; a parallelogram is compared with a rectangle of the same area by cutting and pasting a right triangle on the parallelogram).

2.1 Measure, identify, and draw angles, perpendicular and parallel lines, rectangles, and triangles by using appropriate tools (e.g., straightedge, ruler, compass, protractor, drawing software).

2.2 Know that the sum of the angles of any triangle is 180° and the sum of the angles of any quadrilateral is 360° and use this information to solve problems.

Students need to *commit to memory* the formulas for the area of a triangle, a parallelogram, and a rectangle and the volume of a rectangular solid.

Students need to *commit to memory* the formulas for the area of a triangle, a parallelogram, and a rectangle and the volume of a rectangular solid.

The statement that the sum of the angles of a triangle is 180° is one of the basic facts of plane geometry, but for students in grade five, convincing them of this fact through direct measurements is more important than giving a proof.

Statistics, Data Analysis, and Probability

The ability to graph functions is an essential fundamental skill, and there is no doubt that linear functions are the most important for applications of mathematics. As a result, the importance of these topics can hardly be overestimated.

Closely related to these standards are the following two standards from the Statistics, Data Analysis, and Probability strand:

1.4 Identify ordered pairs of data from a graph and interpret the meaning of the data in terms of the situation depicted by the graph.

1.5 Know how to write ordered pairs correctly; for example, (x, y).

These standards indicate the ways in which the skills involved in the Algebra and Functions strand can be reinforced and applied.

Considerations for Grade-Level Accomplishments in Grade Five

At the beginning of grade five, students need to be assessed carefully on their knowledge of the core content taught in the lower grades, particularly in the following areas:

At the beginning of grade five, students need to be assessed carefully on their knowledge of the core content taught in the lower grades.

- Knowledge and fluency of basic fact recall, including addition, subtraction, multiplication, and division facts (By this level, students should know all the basic facts and be able to recall them instantly.)
- Mental addition—The ability to mentally add a single-digit number to a two-digit number
- Rounding off numbers in the hundreds and thousands to the nearest ten, hundred, or thousand and rounding off two-place decimals to the nearest tenth
- Place value—The ability to read and write numbers through the millions
- Knowledge of measurement equivalencies, both customary and metric, for time, length, weight, and liquid capacity
- Knowledge of prime numbers and the ability to determine prime factors of numbers up to 50
- Ability to use algorithms to add and subtract whole numbers, multiply a two-digit number and a multidigit number, and divide a multidigit number by a single-digit number
- Knowledge of customary and metric units and equivalencies for time, length, weight, and capacity

All of the topics listed previously need to be taught over an extended period of time. A systematic program must be established to enable students to reach high rates of accuracy and fluency with these skills.

Important mathematical skills and concepts for students in grade five to acquire are as follows:

- **Understanding long division.** Long division requires the application of a number of component skills. Students must be able to round tens and hundreds numbers and work estimation problems, divide a two-digit number

Grade Five

Students often become confused with operations with negative numbers because too much is introduced at once.

into a two- or three-digit number mentally and with paper and pencil, and do the steps in the division algorithm. For grade five it suffices to concentrate on problems in which the estimations give the correct numbers in the quotient. This algorithm needs to be taught efficiently so that excessive amounts of instructional time are not required.

- **Adding and subtracting fractions with unlike denominators.** See the instructional profile (Appendix A) on adding and subtracting fractions with unlike denominators.
- **Working with negative numbers.** The standards call for students to add and subtract negative numbers. Students must be totally fluent with these two operations. Students often become confused with operations with negative numbers because too much is introduced at once, and they do not have the opportunity to master one type before another type is introduced. This material must be presented carefully.
- **Ordering fractions and decimal numbers.** Students can use fraction equivalence skills for comparing fractions and for converting fractions to decimals. Students need to know that $\frac{3}{4} = \frac{75}{100} = 0.75 = 75\%$.
- **Working with percents.** To compute a given percent of a number, students can convert the percent to a decimal and then multiply. Students must know that 6% translates to 0.06 (percents under ten percent can be troublesome). Students should be assessed on their ability to multiply decimals by whole numbers before work begins on this type of problem.

Grade Six Areas of Emphasis

By the end of grade six, students have mastered the four arithmetic operations with whole numbers, positive fractions, positive decimals, and positive and negative integers; they accurately compute and solve problems. They apply their knowledge to statistics and probability. Students understand the concepts of mean, median, and mode of data sets and how to calculate the range. They analyze data and sampling processes for possible bias and misleading conclusions; they use addition and multiplication of fractions routinely to calculate the probabilities for compound events. Students conceptually understand and work with ratios and proportions; they compute percentages (e.g., tax, tips, interest). Students know about π and the formulas for the circumference and area of a circle. They use letters for numbers in formulas involving geometric shapes and in ratios to represent an unknown part of an expression. They solve one-step linear equations.

Number Sense

1.0 1.1 1.2 1.3 1.4
2.0 2.1 2.2 2.3 2.4

Algebra and Functions

1.0 1.1 1.2 1.3 1.4
2.0 2.1 2.2 2.3
3.0 3.1 3.2

Measurement and Geometry

1.0 1.1 1.2 1.3
2.0 2.1 2.2 2.3

Statistics, Data Analysis, and Probability

1.0 1.1 1.2 1.3 1.4
2.0 2.1 2.2 2.3 2.4 2.5
3.0 3.1 3.2 3.3 3.4 3.5

Mathematical Reasoning

1.0 1.1 1.2 1.3
2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7
3.0 3.1 3.2 3.3

Key Standards and Elaboration

Number Sense

Grade Six

Most of the standards in the Number Sense strand for the sixth grade are very important. These standards can be organized into four groups. The first is the comparison and ordering of positive and negative fractions (i.e., rational numbers), decimals, or mixed numbers and their placement on the number line:

1.1 Compare and order positive and negative fractions, decimals, and mixed numbers and place them on a number line.

The ordering of fractions is best done through the use of the *cross-multiplication algorithm.*

The ordering of fractions is best done through the use of the *cross-multiplication algorithm*, which says $\frac{a}{b} = \frac{c}{d}$ exactly when $ad = bc$, and $\frac{a}{b} < \frac{c}{d}$ exactly when $ad < bc$. Students not only must be fluent in the use of this algorithm but also must *understand why it is true*. The reason for the latter goes back to the previous observation in the sections for grades four and five that any two fractions can be rewritten as two fractions with the same denominator. Thus $\frac{a}{b}$ and $\frac{c}{d}$ can be rewritten as $\frac{ad}{bd}$ and $\frac{bc}{bd}$. The cross-multiplication algorithm now becomes obvious.

Of particular importance is the students' understanding of the positions of the negative numbers and the geometric effect on the numbers of the number line when a number is added or subtracted from them.

The second group is represented by the next three standards, all of which refer to ratios and percents:

1.2 Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) to show the relative sizes of two quantities, using appropriate notations (a/b, a to b, a:b).

1.3 Use proportions to solve problems (e.g., determine the value of N if $\frac{4}{7} = \frac{N}{21}$, find the length of a side of a polygon similar to a known polygon). Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

1.4 Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

Although Standards 1.2 and 1.3 precede Standard 2.1, they need to be taught after students know all about Standard 2.1; that is, after they have learned about the multiplication and division of fractions. (An example of the need to follow this order is that Standard 1.3 explicitly uses the language of "multiplicative inverse"). Once students have learned these concepts, they can be taught the definition of a ratio as the division of one number by another; for example, the ratio of miles traveled to hours traveled (miles per hour), the ratio of the weights of two bags of potatoes, and so forth. While presenting Standard 1.4, the teacher must be sure to explain why the concept of *percent* is useful: it standardizes the comparison of magnitudes and, in most situations, facilitates computations. For example, one can imagine the confusion that would arise if the sales tax of one state is $\frac{17}{200}$ and that of another state is $\frac{4}{45}$. Which state has a higher sales tax? By agreeing to express the tax as a percent, the two states would normalize their taxes to approximately 8.5% and 8.9%, respectively. Then one can tell at a glance that the second tax rate is higher. Of course, the expression in terms of percent makes the computation of sales tax relatively easy: an 8.5% tax on an article costing \$25.50 is $25.50 \times 0.085 = \$2.17$.

The third group includes the remaining Number Sense standards, all of which relate to fractions:

2.0 Students calculate and solve problems involving addition, subtraction, multiplication, and division.

Because of the slight ambiguity of the language in Standard 2.0, the teacher should clarify that *this standard deals with the four arithmetic operations of positive fractions and with positive and negative integers*. The arithmetic operations of all rational numbers, that is, positive and negative fractions, are left to grade seven.

Since the addition and subtraction of fractions have been taught in grade five (Number Sense Standard 2.3), the main emphasis of sub-Standards 2.1 and 2.2 is on the multiplication and division of positive fractions. A common mistake is to launch immediately into the formula $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ without first giving meaning to the product of fractions $\frac{a}{b} \times \frac{c}{d}$. One can define the fractions $\frac{a}{b} \times \frac{c}{d}$ as the area of a rectangle with side lengths $\frac{a}{b}$ and $\frac{c}{d}$ (in which case the whole of which the product measures a part is the area of the unit square) or as the fraction which is a parts of $\frac{c}{d}$ when $\frac{c}{d}$ is divided into b equal parts. Both interpretations are useful in problem solving, and the relationship between the two should be clearly explained.

From the explanation of grade five Standard 2.4 (Number Sense) in this chapter, the division of fractions is now straightforward: the expression

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{m}{n}$$

means the same thing as $\frac{a}{b} = \frac{m}{n} \times \frac{c}{d}$. From grade four Standard 2.2 (Algebra and Functions), students know that the equation will hold if both sides are multiplied by $\frac{d}{c}$; and therefore, $\frac{a}{b} \times \frac{d}{c} = \frac{m}{n} \times \frac{c}{d} \times \frac{d}{c}$. The product of the last two fractions is just 1, so $\frac{m}{n} = \frac{a}{b} \times \frac{d}{c}$, and the invert-and-multiply rule for division of fractions is shown to be valid.

Standard 2.1 calls for solving problems that make use of multiplication and division of fractions. It is important that students know why the invert-and-multiply rule is sufficient for these applications.

It was mentioned in the section for grade five in this chapter that the concept of least common multiple plays a role in the teaching of fractions. The following standard makes this point explicit:

2.4 Determine the least common multiple and the greatest common divisor of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions or to find the reduced form for a fraction).

The use of the least common multiple (LCM) in fractions should be carefully qualified. On the one hand, a knowledge of LCM does lead to simplifications in some situations; for example,

$$\frac{3}{16}-\frac{1}{24}=\frac{(3\times 3)-(1\times 2)}{48}=\frac{7}{48},$$

in which the LCM of 16 and 24 is 48. Using the LCM is obviously simpler than using the denominator 16×24. On the other hand, finding the LCM of the denominators can be computationally intensive. For example, is it faster, when adding $\frac{2}{57}+\frac{3}{95}$, to determine the LCM of the denominators (which is 285) or simply to use their product as a common denominator?

Grade Six

$$\frac{2}{57}+\frac{3}{95} \text{ as } \frac{(2\times 95)+(3\times 57)}{57\times 95}=\frac{361}{57\times 95}=\frac{361}{5415}$$

Reducing the fraction $\frac{361}{5415}$ to $\frac{1}{15}$ may be more difficult than finding the LCM first and then reducing $\frac{19}{285}$ to $\frac{1}{15}$. Therefore, the decision on whether to use the LCM should be based on an estimate of which method is more straightforward and whether there is a need to generate a reduced form of the sum.

The fourth group stands alone because it consists of only one standard:

2.3 Solve addition, subtraction, multiplication, and division problems, including those arising in concrete situations, that use positive and negative integers and combinations of these operations.

For the first time, students are asked to be completely fluent with the arithmetic of negative integers.

For the first time, students are asked to be completely fluent with the arithmetic of negative integers. Students find this concept difficult because the reasons for some of the more basic rules seem obscure to them. The addition of positive integers may not be an issue, but if one of a and b is negative in $a + b$, then how should a student evaluate this sum? The most important thing to remember is that for any integer a, $-a$ is the number satisfying $a + (-a) = 0$. Students can now see how to add two negative numbers:

$$(-3) + (-5) = -(3 + 5),$$

because the number $[(-3) + (-5)]$ satisfies $[(-3) + (-5)] + (3 + 5) = (-3) + 3 + (-5) + 5 = 0 + 0 = 0$ (where the associative and commutative laws were employed), so that $[(-3) + (-5)] + [3 + 5] = 0$, which means $[(-3) + (-5)] = -(3 + 5)$. In general, if a and b are positive integers, then

$$(-a) + (-b) = -(a + b).$$

This is because $[(-a) + (-b)] + (a + b) = (-a) + a + (-b) + b = 0 + 0 = 0$ (where again the associative and commutative laws were used), so that $[(-a) + (-b)] + (a + b) = 0$, which then implies $(-a) + (-b) = -(a + b)$. If a and b are positive integers and $a < b$, then $a + (-b)$ can be computed in the following way: let c be a positive integer so that $a + c = b$, then

$$a + (-b) = -c.$$

Here is why. It has just been shown that $-b = -(a + c) = (-a) + (-c)$, and so $a + (-b) = a + (-a) + (-c) = 0 + (-c) = -c$, as claimed. In like manner, it can be shown that if $a + c = b$ for positive integers a, b, c, then

$$(-a) + b = c,$$

because $(-a) + b = (-a) + a + c = c$. This explanation shows how to add any two integers.

The multiplication of integers is discussed next. We first observe that $(-3) \times 5 = -(3 \times 5)$. It is sufficient to show, by the usual reasoning, that $[(-3) \times 5] + [3 \times 5] = 0$. This is so because we make use of the distributive law and obtain $[(-3) \times 5] + [3 \times 5] = [(-3) + 3] \times 5 = 0 \times 5 = 0$. More generally, and by the same reasoning, if a and b are any two integers, then

$$(-a) \times b = -(a \times b).$$

It similarly follows that $(-a) \times (-b) = -(a \times (-b)) = -(-(a \times b)) = (-1 \times -1) \times (a \times b)$. It remains to be shown that

$$(-1) \times (-1) = 1.$$

It is enough to show that $\{(-1) \times (-1)\} + (-1) = 0$ because a number that gives 0 when added to (-1) must be 1. By the distributive law, $\{(-1) \times (-1)\} + (-1) = \{(-1) \times (-1)\} + \{(-1) \times 1\} = (-1) \times \{(-1) + 1\} = (-1) \times 0 = 0$, which is to be proved. To sum up, $(-a) \times (-b) = (-1 \times -1) \times (a \times b) = 1 \times (a \times b) = a \times b$.

Algebra and Functions

In the Algebra and Functions strand, the important standards are 1.1 and 2.2. The standard that follows is an expansion of the discussion of linear equations that was begun in the fifth grade:

1.1 Write and solve one-step linear equations in one variable.

Students in the sixth grade should understand and be able to solve simple one-variable equations that are critically important for all applied areas of mathematics. At a more advanced grade level, students will be required to solve systems of linear equations. In grade six they should be able to justify each step in evaluating linear equations as cited in Standard 1.3 (Algebra and Functions). This skill is critical to the algebraic reasoning that is to follow and to the development of carefully applied logic at each step of the process.

Standard 1.1 is closely related to the standards for ratio and percent in the Number Sense strand (Standards 1.2 and 1.4).

Rate and ratio are merely different interpretations in different contexts of dividing one number by another.

2.2 Demonstrate an understanding that *rate* is a measure of one quantity per unit value of another quantity.

Standard 2.2 emphasizes the importance of *understanding* the meaning of the concepts of rate and ratio. Rate and ratio are merely different interpretations in different contexts of dividing one number by another. This standard is also closely related to the problems of rates, average speed, distance, and time that are introduced in Standard 2.3.

Grade Six

Measurement and Geometry

The following core standards are a part of the Measurement and Geometry strand:

1.1 Understand the concept of a constant such as π; know the formulas for the circumference and area of a circle.

2.2 Use the properties of complementary and supplementary angles and the sum of the angles of a triangle to solve problems involving an unknown angle.

Students should know that the volumes of three-dimensional figures can often be found by dividing and combining them into figures whose volumes are already known.

One can define π in many different ways. The recommendation here is to define it as the area of the unit circle rather than as the ratio of the circumference to diameter. The latter is built on *two* concepts relatively new to students, *ratio* and *length of a curve* (circumference), whereas the former uses only the concept of area. Moreover, the area of the unit circle can be approximated directly by the use of (good) grid papers, and students have a good chance of getting $\pi = 3.14 \pm 0.05$. This demonstration would not only create a strong impression on students but also deepen their understanding of both the number π and the concept of area.

Standard 1.3 is also important, and students should know that the volumes of three-dimensional figures can often be found by dividing and combining them into figures whose volumes are already known.

Statistics, Data Analysis, and Probability

The study of statistics is more important in the sixth grade than in the earlier grades. One of the major objectives of studying this topic in the sixth grade is to give students some tools to help them understand the uses and misuses of statistics. The core standards for Statistics, Data Analysis, and Probability that focus on these goals are:

2.2 Identify different ways of selecting a sample (e.g., convenience sampling, responses to a survey, random sampling) and which method makes a sample more representative for a population.

2.3 Analyze data displays and explain why the way in which the question was asked might have influenced the results obtained and why the way in which the results were displayed might have influenced the conclusions reached.

2.4 Identify data that represent sampling errors and explain why the sample (and the display) might be biased.

2.5 Identify claims based on statistical data and, in simple cases, evaluate the validity of the claims.

For example, if a study of computer use is focused solely on students from Fresno, the class might try to determine how valid the conclusions might be for the students in the entire state. Again, how valid would the conclusion of a study that interviewed 23 teachers from all over the state be for all the teachers in the state? These questions represent major applications of the type of precise and critical thinking that mathematics is supposed to facilitate in students.

In the sixth grade, students are also expected to become familiar with some of the more sophisticated aspects of probability. They start with the following standard:

Grade Six

3.1 Represent all possible outcomes for compound events in an organized way (e.g., tables, grids, tree diagrams) and express the theoretical probability of each outcome.

This strand is challenging but vitally important, not only for its use in statistics and probability but also as an illustration of the power of attacking problems systematically.

The concepts in probability Standards 3.3 and 3.5 may be difficult for students to understand:

3.3 Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, 1-P is the probability of an event not occurring.

3.5 Understand the difference between independent and dependent events.

The topics in both standards need to be carefully introduced, and the terms must be defined. Both the concept that probabilities are measures of the likelihood that events might occur (numerical values for probabilities are usually expressed as numbers between 0 and 1) and the distinction between dependent and independent events are important for students to understand. If students can grasp the meaning of the terms, they can understand the basic points of these standards. This knowledge can help students reach accurate conclusions about statistical data.

The concept that probabilities are measures of the likelihood that events might occur and the distinction between dependent and independent events are important for students to understand.

Considerations for Grade-Level Accomplishments in Grade Six

At the beginning of grade six, students need to be assessed carefully on their knowledge of the core content taught in the early grades, which is described at the beginning of the section for grade five, and on the following content from grade five:

– Increased fluency with the long-division algorithm

– Conversion of percents, decimals, and fractions, including examples that represent a value over 1 (e.g., $2.75 = 2\frac{3}{4} = 275\%$)

– Use of exponents to show the multiples of a single factor

– Addition, subtraction, multiplication, and division with decimal numbers and negative numbers

– Addition of fractions with unlike denominators and multiplication and division of fractions

All of these topics require teaching over an extended period of time. A systematic program must be established so that students can reach high rates of accuracy and fluency with these skills.

Grade Six

All topics delineated in the grade six standards, and in particular the key strands, should be assessed regularly throughout the sixth grade. Once the skills have been taught and mastery demonstrated through assessment, teachers need to continue to review and maintain the students' skills. Mental mathematics, warm-up activities, and additional questions on tests can be used to accomplish this task.

Important mathematical skills and concepts for students in grade six to acquire are as follows:

- **The least common multiple and the greatest common divisor.** Students can become confused by the concepts of the least common multiple (LCM) and the greatest common divisor (GCD). The least common multiple of two numbers includes examples in which one multiple is in fact the least common multiple (e.g., 2 and 8; the LCM is 8); the least common multiple is the product of the two numbers (e.g., 4 and 5; the LCM is 20); and the least common multiple is a number that fits into neither of the two first categories (6 and 8; the LCM is 24). The teaching sequence should include examples of all three types. Finding the LCM becomes much more difficult with large numbers (e.g., finding the LCM of 36 and 48). One way to determine the answers is with prime factors, $36 = 2 \times 2 \times 3 \times 3$ and $48 = 2 \times 2 \times 2 \times 2 \times 3$. The LCM is $2 \times 2 \times 2 \times 2 \times 3 \times 3$, or 144. The process for finding the LCM can be confused with the process for finding the greatest common divisor (what is the GCD of 12 and 16?) because both deal with multiples of prime factors of numbers. Students should also be told that when a number is very large (e.g., 250 digits), finding its prime factorization is impractical, even with the help of the most powerful computers now available. There are other methods besides finding their prime factorization to determine the GCD and LCM.
- **Discounts, interest, and tips.** Within this realm are problems that range from simple one-step problems to more complex multistep problems. Programs must be organized so that easier problems are introduced first, followed by a thorough teaching of significantly more difficult problems. An example of a simple discount problem is, *A dress cost 50 dollars. There is a 10 percent discount. How many dollars will the discount be?* This problem is solved by performing the calculation for 10 percent of 50. If the problem asks, *How much will the dress cost with the discount?* the students would have to subtract the discount from the original price. A much more complex problem would

be, *The sale price of a dress is 40 dollars. The discount was 20 percent. What was the original cost of the dress?* The problem might be solved through several procedures, all of which would involve the application of many more skills than those called for in the first problem. To work the third problem, the student has to know that the original price equates with 100 percent and the sales price is 80 percent of the original price. One way of solving the problem is for the student to write the equation 0.80 $N = 40$, with N representing the original price. Thus $N = \frac{40}{0.80} = 50$. This way of solving the problem focuses on the increased emphasis on the use of variables in the Algebra and Functions strand. The computation skills needed to solve for N obviously need to be taught before this type of problem is introduced.

The treatment of interest at this grade is meant to deal with simple interest in one accrual period. It is not intended to extend to compound interest over several accrual periods in which the time is expressed as an exponent, as is the case for the normal computation formula for compound interest.

- **Multiplication and division of fractions.** Students should learn why and how fractions are multiplied and divided. Students must understand why the second fraction in a division problem is inverted, if that process is used. Students need to know when to use multiplication or division in application problems. For example, *There are 24 students in our class. Two-thirds of them passed the test. How many students passed the test?* is solved through multiplying; while the problem, *A piece of cloth that is 12 inches long is going to be cut into strips that are $\frac{2}{3}$ of an inch long. How many strips can be made?* is solved through division. Structured systematic teaching must be done to help students determine which procedure to use in solving different problems.

Students must understand why the second fraction in a division problem is inverted, if that process is used.

Grade Seven Areas of Emphasis

By the end of grade seven, students are adept at manipulating numbers and equations and understand the general principles at work. Students understand and use factoring of numerators and denominators and properties of exponents. They know the Pythagorean theorem and solve problems in which they compute the length of an unknown side. Students know how to compute the surface area and volume of basic three-dimensional objects and understand how area and volume change with a change in scale. Students make conversions between different units of measurement. They know and use different representations of fractional numbers (fractions, decimals, and percents) and are proficient at changing from one to another. They increase their facility with ratio and proportion, compute percents of increase and decrease, and compute simple and compound interest. They graph linear functions and understand the idea of slope and its relation to ratio.

Number Sense

1.0	1.1	**1.2**	1.3	**1.4**	**1.5**	1.6	**1.7**
2.0	2.1	**2.2**	**2.3**	2.4	**2.5**		

Algebra and Functions

1.0	1.1	1.2	**1.3**	1.4	1.5
2.0	2.1	2.2			
3.0	3.1	3.2	**3.3**	**3.4**	
4.0	**4.1**	**4.2**			

Measurement and Geometry

1.0	1.1	1.2	**1.3**			
2.0	2.1	2.2	2.3	2.4		
3.0	3.1	3.2	**3.3**	**3.4**	3.5	**3.6**

Statistics, Data Analysis, and Probability

1.0	1.1	1.2	**1.3**

Mathematical Reasoning

1.0	1.1	1.2	1.3					
2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8
3.0	3.1	3.2	3.3					

Key Standards and Elaboration

Number Sense

The first basic standard for the Number Sense strand is:

1.2 Add, subtract, multiply, and divide rational numbers (integers, fractions, and terminating decimals) and take positive rational numbers to whole-number powers.

At this point the students should understand arithmetic involving rational numbers. Negative fractions are formally introduced and studied for the first time. Students should know the difference between rational and irrational numbers (Standard 1.4) and be aware that numbers such as the square root of two are not rational. Here, teachers should take care not to misinform the students. For example, some textbooks assert that the square root of 2 is not a rational number and then "prove" that assertion by producing a calculator-generated representation of $\sqrt{2}$ to perhaps 15 decimal places and state that the decimal is not repeating. That is unacceptable. It is better to use the facts in the standard (Standard 1.5) to construct an explicit nonrepeating decimal:

Negative fractions are formally introduced and studied for the first time.

1.5 Know that every rational number is either a terminating or a repeating decimal and be able to convert terminating decimals into reduced fractions.

One can construct a nonrepeating decimal, for example, by putting zeros in all the places past the decimal point except for putting ones in (1) the first, second, fourth, and eighth places and, generally, the places marked by each power of 2:

$$0.11010001000000010000000000000001000 0\ldots;$$

or perhaps (2) the first, third, sixth, tenth, and generally, the places marked by $\frac{n(n+1)}{2}$:

$$0.1010010001000010000010000000100\ldots.$$

In this way students will see how to construct vast quantities of irrational numbers. At this point it might be possible to challenge the advanced students by showing them that a specific number (such as $\sqrt{2}$) is, in fact, irrational. They then can learn that while there are vast quantities of both rational and irrational numbers, it is often very difficult to show that specific numbers are in one set or the other. But this sophisticated material should not be emphasized for the class as a whole. In particular, at this stage it is probably not wise to attempt any kind of a proof of the facts in Standard 1.5. The students can be told that this basic awareness of irrationality is sufficiently important to be discussed at this point even though its justification will have to be deferred until they take a more advanced course.

By now the students should have enough skill with factoring integers so that they can use factoring to find the smallest common multiple of two whole numbers (Standard 2.2). Teachers should emphasize, once again, that the correct formula for the sum of two fractions is

$$\left(\frac{a}{b}\right)+\left(\frac{c}{d}\right)=\frac{(ad+bc)}{bd}$$

and that the usual algorithm using factoring to find the smallest common denominator is but a refinement of this formula. (See the discussion of Number Sense Standard 2.2 for the fifth grade.) For the purpose of finding smallest common denominators, students should become more familiar with the basic exponent rules (Standard 2.3), which will have direct applications in the main seventh grade application of compound interest.

The last topic in the first standard of the Number Sense strand (Standard 1.7) is also one of the high points of the entire strand:

1.7 Solve problems that involve discounts, markups, commissions, and profit and compute simple and compound interest.

Computing interest is a skill that can mean the difference between students managing their money and other resources well or not at all.

This is a major topic, which should come toward the end of the year and should be a major highlight of the kindergarten through grade seven mathematical experience. It provides one of the most important applications of mathematics in students' everyday life, a skill that can mean the difference between students managing their money and other resources well or not at all. Mastery of this standard requires a good grasp of the concept of percent, the laws of exponents, and the distributive law.

Standard 2.5, the last standard in the Number Sense strand, on absolute value should receive some emphasis. This topic is usually slighted in middle schools and high schools; however, students should acquire some facility with this concept as early as possible. The students need to understand that the correct way to express the statement "two numbers x and y are close to each other" is "$|x–y|$ is small." The concept of two numbers being "close" was introduced in grade four in connection with rounding off (see "Elaboration" in grade four).

Algebra and Functions

Familiarity with the distributive law, the associative law, and the commutative rule for addition and multiplication of whole numbers has been mentioned at several points previously in the Algebra and Functions standards in grades five and six. For these standards in grade seven, the concepts are taken a step further with the following:

1.3 Simplify numerical expressions by applying the properties of rational numbers (e.g., identity, inverse, distributive, associative, commutative) and justify the process used.

This is a critical step because it shows the power of abstract thinking in helping to make sense of complex situations and to derive the basic properties of rational numbers.

One of the most basic topics in applications of mathematics is systems of linear equations.

One of the most basic topics in applications of mathematics is systems of linear equations. A clear understanding of even something as simple as systems of two linear equations in two unknowns is crucial to understanding more advanced topics, such as calculus and analysis. The first major steps are taken toward this goal when the study of a single linear equation is initiated in these four standards:

3.3 Graph linear functions, noting that the vertical change (change in *y*-value) per unit of horizontal change (change in *x*-value) is always the same and know that the ratio ("rise over run") is called the slope of a graph.

3.4 Plot the values of quantities whose ratios are always the same (e.g., cost to the number of an item, feet to inches, circumference to diameter of a circle). Fit a line to the plot and understand that the slope of the line equals the ratio of the quantities.

4.1 Solve two-step linear equations and inequalities in one variable over the rational numbers, interpret the solution or solutions in the context from which they arose, and verify the reasonableness of the results.

4.2 Solve multistep problems involving rate, average speed, distance, and time or a direct variation.

Again, the connection of the second standard with the Measurement and Geometry Standard 1.3 should be noted. These topics provide excellent problems to test the students' understanding of the techniques for solving linear equations.

Students at this stage of algebraic development should be able to understand a clarification of the somewhat subtle concepts of *ratio* and *direct proportion* (sometimes called *direct variation*). The "ratio between two quantities" is nothing more or less than a particular interpretation of "one quantity divided by another in the sense of numbers." Of course, thus far students know only how to divide rational numbers. The teacher should tell the students that the division between irrational numbers will also be explained to them in more advanced courses; therefore, this definition of *ratio* will still apply. *Direct variation* can be explained in terms of linear functions: "A varies directly with B" means that "for a fixed constant c, $A = cB$." Teachers and textbooks commonly try to "explain" the meanings of both terms in abstruse language, resulting in confusion among students and even teachers. No explanation is necessary: *ratio* and *direct variation* are mathematical terms, and they should be clearly defined once the students have been taught the necessary facts and techniques.

Grade Seven

Measurement and Geometry

The first major emphasis in the Measurement and Geometry strand is for the students to develop an increased sense of spatial relations. This topic is reflected in these two standards:

3.4 Demonstrate an understanding of conditions that indicate two geometrical figures are congruent and what congruence means about the relationships between the sides and angles of the two figures.

3.6 Identify elements of three-dimensional geometric objects (e.g., diagonals of rectangular solids) and describe how two or more objects are related in space (e.g., skew lines, the possible ways three planes might intersect).

A critical part of understanding this material is that the students know the general definition of *congruence*—two figures are congruent if a succession of reflections, rotations, and translations will make one coincide with the other—and understand that properties of congruent figures, such as angles, edge lengths, areas, and volumes, are equal. The concepts of reflections, rotations, and translations in the plane can be made more accessible by tracing identical geometric figures on two transparencies and then allowing one to move against the other.

The next basic step is contained in the following standard:

3.3 Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement.

The Pythagorean theorem is probably the first true theorem that the students will have seen. It should be emphasized that students are not expected to prove this result. But the better students should be able to understand the proof given by cutting, in two different ways, a square with the edges of length $a + b$ (where a and b are the lengths of the legs of the right triangle). However, everyone is expected to understand what the theorem and its converse mean and how to use both. The applications can include understanding the formula that the square root of $x^2 + y^2$ is the length of the line segment from the origin to the point (x, y) in the plane and that the shortest distance from a point to a line not containing the point is the length of the line segment from the point perpendicular to the line.

Seventh grade students should memorize the formulas for the volumes of cylinders and prisms.

Although the following topics are not as basic as the preceding ones, they should also be covered carefully. Seventh grade students should memorize the formulas for the volumes of cylinders and prisms (Standard 2.1). Students at this point should understand the discussion that began in the sixth grade concerning the volume of "generalized cylinders." More precisely, they should think of a right circular cylinder as the solid traced by a circular disc as this disc moves up a line segment *L* perpendicular to the disc itself. More generally, the disc is replaced with a planar region of any shape, and the line segment *L* is no longer required to be perpendicular to the planar region. Then, as the planar region moves up along *L,* always parallel to itself, it traces out a solid called a generalized cylinder. The formula for the volume of such a solid is still (height of the generalized cylinder) × (area of the planar region). *Height* now refers to the vertical distance between the top and bottom of the generalized cylinder.

The final topic to be emphasized in seventh grade Measurement and Geometry is as follows:

1.3 Use measures expressed as rates (e.g., speed, density) and measures expressed as products (e.g., person-days) to solve problems; check the units of the solutions; and use dimensional analysis to check the reasonableness of the answer.

This standard interacts well with the demands of the algebra standards, particularly in solving linear equations. Typically, the main difficulty in understanding problems of this kind is keeping the definitions and the physical significance of the various measures straight; therefore, care should be taken to emphasize the meanings of the terms in the various problems.

Statistics, Data Analysis, and Probability

The most important of the three seventh grade standards in Statistics, Data Analysis, and Probability is this:

1.3 Understand the meaning of, and be able to compute, the minimum, the lower quartile, the median, the upper quartile, and the maximum of a data set.

These are useful measures that students need to know well. Care should be taken to ensure that all students know the definitions, and many examples should be given to illustrate them.